The Well-Separated Pair Decomposition

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February 2015

7^{th} Winter School on Computational Geometry

Outline

1. Introduction
2. Definition of the Well-Separated Pair Decomposition
3. Computing the Well-Separated Pair Decomposition
4. The split tree
5. Extension to Other Metrics

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- $P =$ set of n points in $\mathbb{R}^d$.
- $D = \{|pq| \ \ |p, q \in P\}$.
- $|D| = ?$
- Decomposition of $P \times P = \{A_i\}_i$ s.t. $P \times P = \cup_i A_i$,
  $A_i =$ pairs with same distance.
- $|\{A_i\}_i| = |D| \in \Theta(n^2)$.
- What if $A_i =$ pairs with almost same distance?
  Is there a decomposition with size $o(n^2)$?
- Answer:

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The Well-Separated Pair Decomposition

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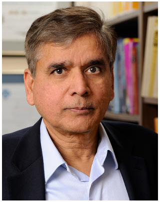
- Answer: YES! Size $\mathcal{O}(n)$ exists!
  Use Well-Separated Pair Decomposition.

Well-Separated Pair Decomposition:

- WSPD: introduced by Callahan and Kosaraju in 1995.
- It applications: To solve a large variety of proximity problems

Paul B. Callahan



S.Rao Kosaraju

Definition of WSPD

Well-Separated Pair

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$A, B \subset \mathbb{R}^d$ are s-well-separated pair ($s > 0$ constant), if $\exists$ disjoint balls, D_A and D_B such that

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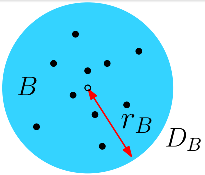

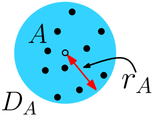

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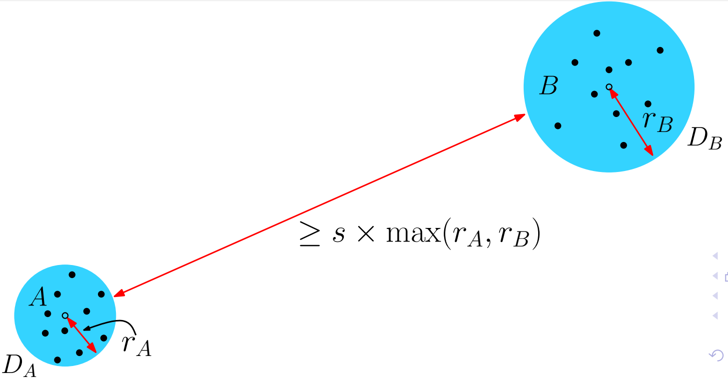
- $A \subseteq D_A$ and $B \subseteq D_B$.
- $\mathbf{d}(D_A, D_B) \geq s \times \max(\text{radius}(D_A), \text{radius}(D_B))$.

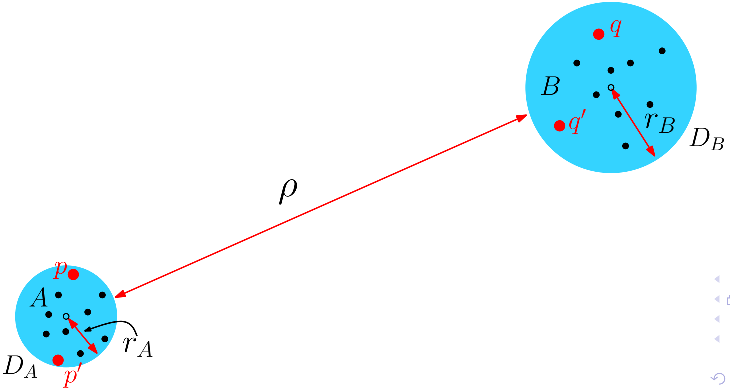


Property of WSP's

Property of Well-Separated Pairs

If (A, B) is a s-well-separated, $p, p' \in A$, $q, q' \in B$, then

- $|pp'| \leq (2/s)|pq|$
- $|p'q'| \leq (1 + 4/s)|pq|$



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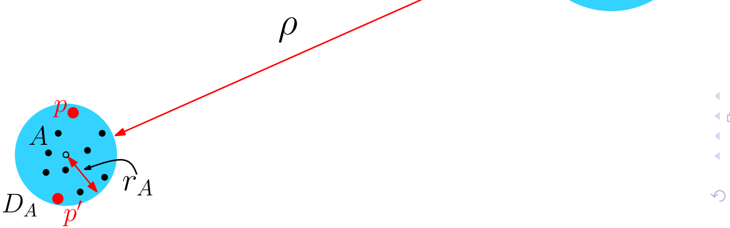
$$\begin{aligned} |pq| &\geq \rho \\ &\geq s \times \max\{r_A, r_B\} \\ &\geq s \times \frac{|pp'|}{2}. \end{aligned}$$



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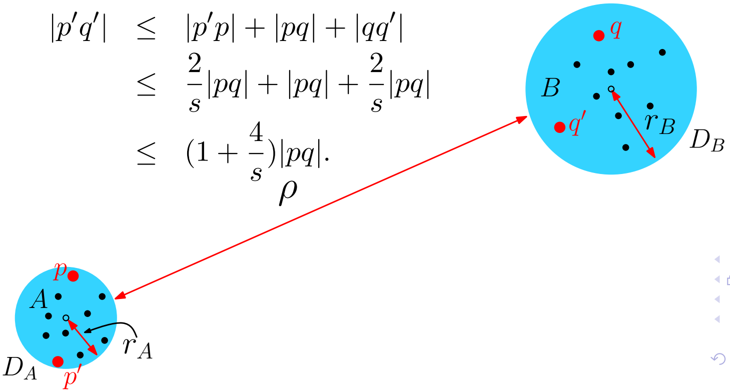

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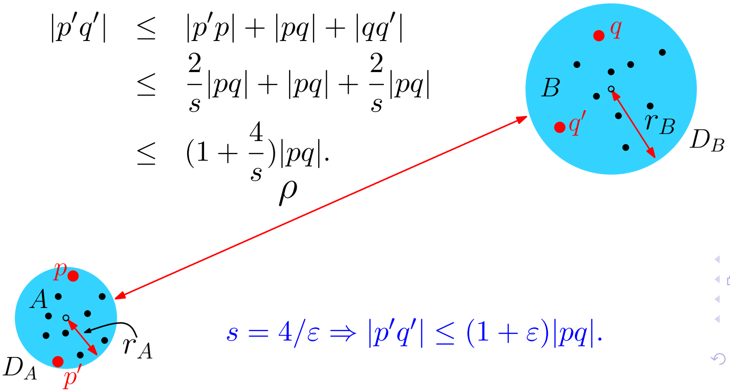

$s = 4/\varepsilon \Rightarrow |p'q'| \leq (1 + \varepsilon)|pq|.$

Well-Separated Pair Decomposition:

Let $P \subset \mathbb{R}^d$ and $s > 0$. A WSPD for P w.r.t. s is a set $\{(A_i, B_i)\}_{i=1}^m$ of pairs of non-empty subsets of V s. t.

- $\forall i$, A_i and B_i are s-well-separated pair,
- $\forall p, q \in V$, there is exactly one index i s. t.
  - $p \in A_i$ and $q \in B_i$ or
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More precisely: $P \times P = \bigcup_{i=1}^{m} (A_i \times B_i)$

m : Size of WSPD.

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WSPD exists?

Question

Does a WSPD exist for any set P?

Answer

Yes. We can consider WSPD for P as follows

$$\{\{p_i\}, \{q_i\}\} \qquad \forall \text{ distinct points } p_i \text{ and } q_i \text{ of } P$$

Size: $\mathcal{O}(n^2)$.

WSPD of size $\mathcal{O}(n)$?

(Callahan & Kosaraju (1995))
For any set of n points, we can construct a WSPD of size $\mathcal{O}(s^d \cdot n)$ in $\mathcal{O}(n \log n)$ time using $\mathcal{O}(s^d \cdot n)$ space.

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How can we find a WSPD of size $\mathcal{O}(n)$ for a set of n points with respect to $s > 0$?

The stages of the algorithms

1. Construct a tree (*split tree* or compressed quad tree).
2. Construct the WSPD using the tree.

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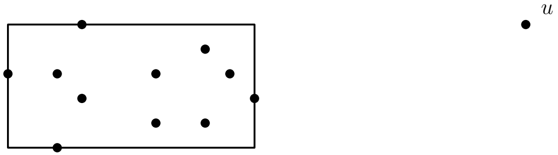
The Split Tree

Main idea:

- Compute bounding box of the points.
- Split the longest edge of the bounding box.
- Recurse on each part (left and right child) if it contains more than 1 point.

The Split Tree

An Example of the Split Tree



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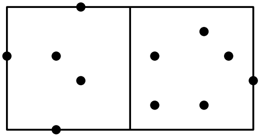

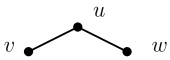

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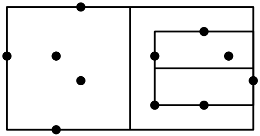

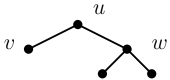

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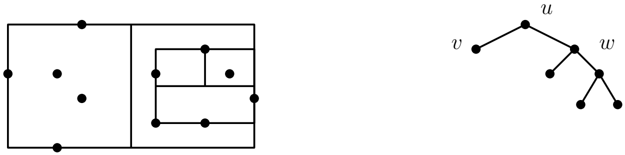

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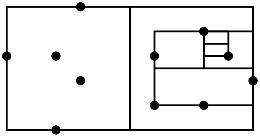

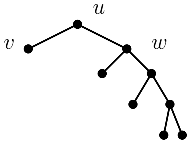

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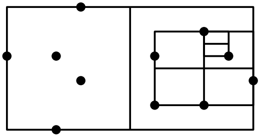

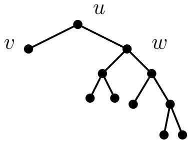

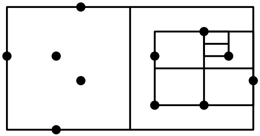

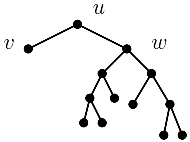
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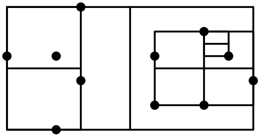

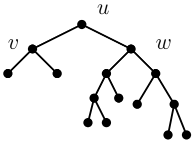
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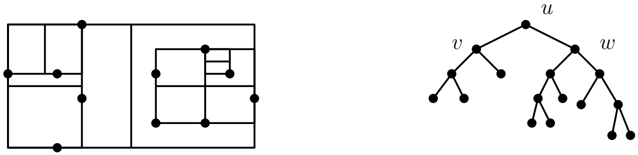

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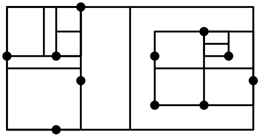

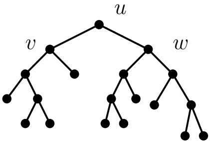

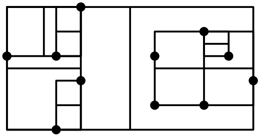

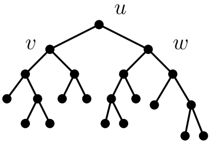

u
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Algorithm SPLITTREE(P, R)

if $|P| = 1$
  then create a new node u;
    $R(u) := R(P)$;
    $R_0(u) := R$;
    store $R(u)$ and $R_0(u)$ with u;
    return node u;
  else compute the bounding box $R(P)$ of P;
    split R into two hyperrectangles R_1 and R_2;
    $P_1 := P \cap R_1$;
    $P_2 := P \setminus P_1$;
    $v :=$ SPLITTREE(P_1, R_1);
    $w :=$ SPLITTREE(P_2, R_2);
    create a new node u;
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Time Complexity: $\Theta(n^2)$ Since the height of tree can be $\Theta(n)$.

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    store $R(u)$ and $R_0(u)$ with u, with children v and w;
    return node u;

Time Complexity: $\Theta(n^2)$ Since the height of tree can be $\Theta(n)$.

But there is an $\mathcal{O}(n \log n)$ implementation for it.

Main Idea:

- Compute Partial Split Tree in $\mathcal{O}(n)$ time.
- Change the Partial Split Tree to Split Tree.

Partial Split Tree

- Same as Split Tree; but leaves can have size between 1 and $n/2$.
- Time needed in each node is proportional to the size of smaller child.
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Computing the WSPD

Computing WSPD

Algorithm COMPUTEWSPD(T, s)
Input: T: Split Tree, $s > 0$.
Output: WSPD for S.

1. **for each** internal node u of T
2. v :=left child of u;
3. w :=right child of u;
4. FINDPAIRS(v,w);

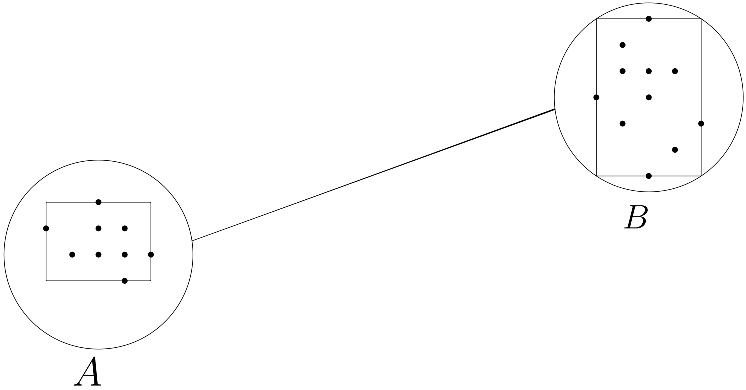
Computing the WSPD

Algorithm FINDPAIRS(v, w)

if P_v and P_w are s-well-separated pair
  then return the node pair $\{v, w\}$
  else if $L_{max}(R(v)) \leq L_{max}(R(w))$
  then
    w_l := left child of w;
    w_r := right child of w;
    FINDPAIRS(v, w_l);
    FINDPAIRS(v, w_r);
  else
    v_l := left child of v;
    v_r := right child of v;
    FINDPAIRS(v_l, w);
    FINDPAIRS(v_r, w);

Bounding boxes is used to decide about the well-separatedness in $\mathcal{O}(1)$ time.

Computing the WSPD



Computing the WSPD

Questions

- FindPairs(v, w) terminates? Yes! Singletons are well-separated.
- Is ComputeWSPD(T, s) correct? Yes! Almost obvious!
- Time complexity of ComputeWSPD(T, s)? $O(m)$, $m = \#$ WSPs.

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Properties of WSPD

A WSPD for P w.r.t. s is a set $\{(A_i, B_i)\}_{i=1}^m$ of pairs of non-empty subsets of P s. t.

- $\forall i$, A_i and B_i are s-well-separated pair,
- $\forall p, q \in P$, there is exactly one index i s. t.
  - $p \in A_i$ and $q \in B_i$ or
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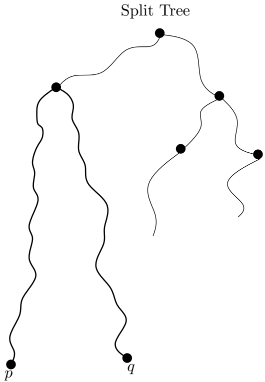


Computing the WSPD
Size of WSPD

Main Idea:

- Make pairs directed.
  - (P_u, P_v) if $(P_{par(u)}, P_v)$ is not WSP.
  - (P_v, P_u) if $(P_u, P_{par(v)})$ is not WSP.
- Each node appears in constant directed pairs as first item (Using Packing Lemma).

Lemma

Let u be any node of the split tree T. There are at most $((2s+4)\sqrt{d}+4)^d$ nodes v in T such that (P_u, P_v) is a directed pair in the WSPD computed by algorithm ComputeWSPD(T, s).

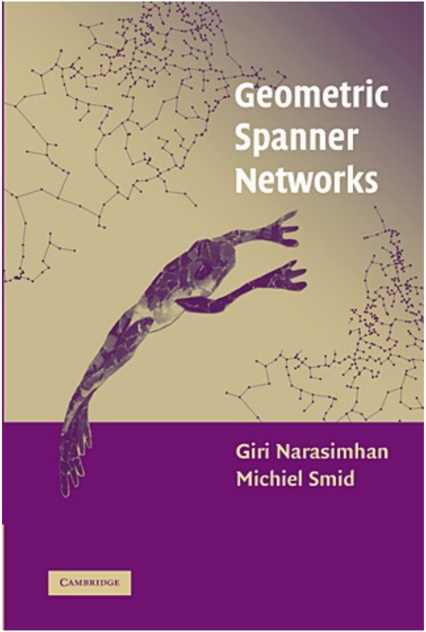
Computing the WSPD

WSPD Theorem

Let $P \subset \mathbb{R}^d$ be a set of n points and $s > 0$.

1. The split tree for P can be computed in $\mathcal{O}(n \log n)$ time.
2. Given the split tree, we can compute in $\mathcal{O}(s^d n)$ time, a WSPD for P with respect to s of size $\mathcal{O}(s^d n)$.

More details ...



WSPD: Applications

- Approximation of the complete graph(Spanners).
- Closest pair, All Nearest Neighbour, k-closest pairs.
- Approximate Euclidean Minimum Spanning Tree.
- ...

WSPD: Total size

Total size of WSPD

There are point sets such that for any WSPD $\{(A_i, B_i)\}_i$ of them,

$$\sum_i (|A_i| + |B_i|) = \Omega(n^2).$$

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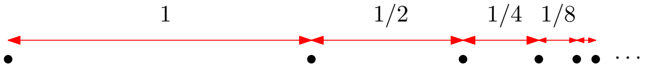

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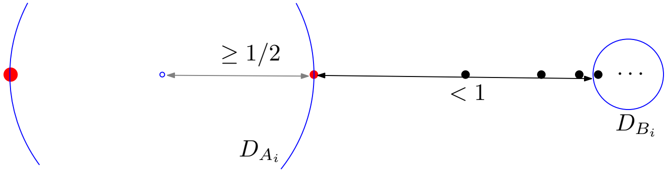

A_i and B_i are not s-well-separated ($s > 2$) because

$$\mathbf{d}(D_{A_i}, D_{B_i}) \not\geq s \times 1/2.$$

So A_i is singleton.

Semi-Separated Pair Decomposition

Semi-Separated Pair:

$A, B \subset \mathbb{R}^d$ are s-semi-separated ($s > 0$), if $\exists$ disjoint balls, D_A and D_B such that

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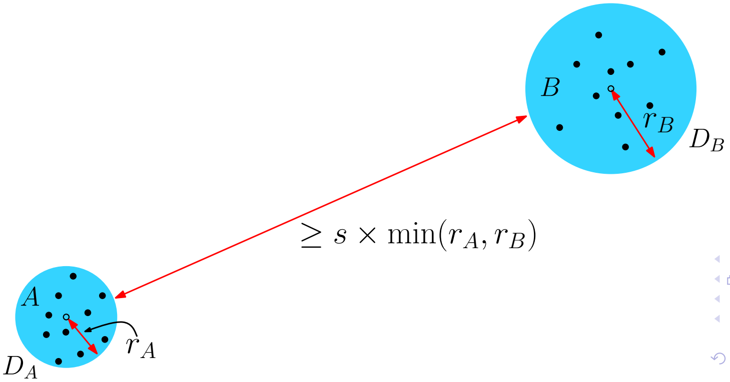
- $A \subseteq D_A$ and $B \subseteq D_B$.
- $\mathbf{d}(D_A, D_B) \geq s \times \min(\text{radius}(D_A), \text{radius}(D_B))$.



Semi-Separated Pair Decomposition:

Just replace WS by SS in WSPD definition.

SSPD Construcion:

For any set P an SSPD $\{(A_i, B_i)\}_{i=1}^{m}$ such that

- $m = \mathcal{O}(s^d \cdot n)$,
- $\sum_i (|A_i| + |B_i|) = \mathcal{O}(s^d \cdot n \log n)$.

can be constructed in near linear time and space.

Note: $\sum_i (|A_i| + |B_i|) = \Omega(n \log n)$.

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Extention to Other Metrics

Metric Space

Metric space: (P, d), $d : P \times P \longrightarrow \mathbb{R}$

1. $d(p,q) \geq 0, \quad \forall p, q \in P$.
2. $d(p,q) = 0$ if and only if $p = q$.
3. $d(p,q) = d(q,p)$.
4. $d(p,q) \geq d(p,r) + d(r,q), \quad \forall p, q, r \in P$.

Diameter and Distance

- *Diameter*

$$D(A) := \max\{d(a,b) : a, b \in A\}.$$

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WSPD in a metric space

(A, B): s-well-separated if

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Open Problem

Which metric spaces (P, d) admit a WSPD of subquadratic size? Design efficient algorithms that compute such a WSPD.

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Thank You.

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