Gap Property

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Introduction

Why do we study, this chapter?

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- Gap property
- Gap theorem ($O(\log n) wt(MST(S))$)

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- A geometric lemma
- The 2-opt algorithm for the traveling salesman problem.

Introduction(Geometric Analysis)

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- So we will follow a two-step approach
  1) Devise a property P
  2) Design an algorithm that computes a spanner whose edge set satisfies property P.

The gap Property

$w \geq 0$, and E be a set of directed edges in R^d

1) W-gap property

$$|pr| > w.\min(|pq|, |rs|)$$

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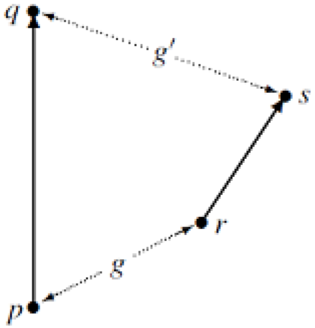

1) W-gap property

$$|pr| > w.\min(|pq|, |rs|)$$

2) Strong w-gap property

$$|pr| > w.\min(|pq|, |rs|)$$

$$|qs| > w.\min(|pq|, |rs|)$$

Gap Theorem

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2) If $w > 0$ then $wt(E) < (1 + \frac{2}{w}).wt(MST(S)). \log n$

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2) If $w > 0$ then $wt(E) < (1 + \frac{2}{w}).wt(MST(S)). \log n$

3) (In case of the strong w-gap property) If $w \geq 0$ then each point of S is the sink of at most one edge of E.

Proof of gap theorem

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$min(|pq|, |rs|) \geq 0$

- W-gap property $|pr| > w. \min(|pq|, |rs|)$
  so $p \neq r$

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- if m($m \geq 3$) is odd, then there is a $E' \in E$ such that $|E'| = \frac{m+1}{2}$ and $wt(E') < 1 + \frac{2}{w} wt(MST(S))$

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if m is even and $E' = \frac{m}{2}$:

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- consider an optimal TSP of S
- Renumber the points of S such that $TSP(S) = (P_1, P_2, ..., P_n, P_1)$

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- Number the edges of E according to the order in which we visit the sources of the edges along this tour as $e_1, e_2, ..., e_m$ (since $S_i's$ are pairwise distinct, this is unique)

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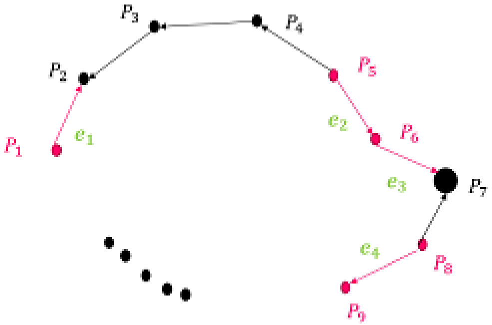
- k_i: The index such that edge e_i has point p_{k_i} as its source $(1 \leq i \leq m)$
  $1 \leq k_1 < k_2 < ... \leq n$



k_1=1

k_2=5

k_3=6

k_4=8

Etc.

$wt(E) < (1 + \frac{2}{w}).wt(MST(S)).\log n$

- for $1 \leq i \leq \frac{m}{2}$ let T_i the portion of $TSP(S)$ that starts at $P_{k_{2i-1}}$ and ends at $P_{K_{2i}}$

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- Ex) $T_1 = (P_1, P_2, P_3, P_4)$

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$$\min(|e_{2i-1}|, |e_{2i}|) < \frac{1}{w} wt(T_i)$$

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- $E'(E' \subset E)$ contains the shorter edge between e_{2i-1} and e_{2i}. $(1 \leq i \leq \frac{m}{2})$

$wt(E) < (1 + \frac{2}{w}).wt(MST(S)).\log n$

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- since $m \leq n$ we'll be done.

$wt(E) < (1 + \frac{2}{w}).wt(MST(S)).\log n$

- $m = 1$

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- $m = 1$
- Now, assume the latter inequality holds for all sets of edges that have less than m elements and satisfy the w-gap property. $m \geq 3$
- let $E' \subset E$ size of E at least $\frac{m}{2}$ and

$$wt(E') < (1 + \frac{2}{w}).wt(MST(S)).\log m$$

$wt(E) < (1 + \frac{2}{w}).wt(MST(S)).\log n$

- size of E/E' is at most $\frac{m}{2}$ and it satisfies the w-gap property.

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- By the induction hypothesis:

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- size of E/E' is at most $\frac{m}{2}$ and it satisfies the w-gap property.
- By the induction hypothesis:
- $wt(E/E') < (1 + \frac{2}{w}).wt(MST(S)).\log \frac{m}{2}$

$wt(E) < (1 + \frac{2}{w}).wt(MST(S)).\log n$

$$wt(E) = wt(E') + wt(E/E') < (1+\frac{2}{w}).wt(MST(S))(1+\log\frac{m}{2})$$

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- It can be used in any other metric space.

A Lower Bound

Theorem 6.2.1:

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- let $n = 3^k - 1$

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$$wt(E) = \Omega(wt(MST(S))) \log n$$

$wt(E) = \Omega(wt(MST(S)) \log n)(proof)$

- We partition the interval $[0, 1]$ into 3^i interval $(0 \leq i < k)$, each having length $\frac{1}{3^i}$.

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- e_{ij};middle of these three subintervals

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$$e_{ij} = [\tfrac{j}{3^i} + \tfrac{1}{3^{i+1}}, \tfrac{j}{3^i} + \tfrac{2}{3^{i+1}}]$$

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$$e_{ij} = [\tfrac{j}{3^i} + \tfrac{1}{3^{i+1}}, \tfrac{j}{3^i} + \tfrac{2}{3^{i+1}}]$$

$$E_i = \{e_{ij} : j = 0, 1, ..., 3^i - 1\}$$

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$$E = E_0 \cup E_1 \cup .... \cup E_{k-1}$$

$wt(E) = \Omega(wt(MST(S))\log n)(proof)$

$|E_i| = 3^i$ so $|E| = \sum 3^i = \frac{3^k - 1}{2}$

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$n = 2 \times |E| = 3^k - 1$

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$$e_{ij} = [\frac{j}{3^i} + \frac{1}{3^{i+1}}, \frac{j}{3^i} + \frac{2}{3^{i+1}}]$$

$$e_{kl} = [\frac{l}{3^l} + \frac{1}{3^{k+1}}, \frac{l}{3^k} + \frac{2}{3^{k+1}}]$$

- if $i = k$ then put $l = j + 1$ to have the nearest sources so we have: $|S_{e_{ij}} - S_{e_{ij+1}}| = \frac{1}{3^{i+1}}$ Done!
- else if $i < k($ or $k < i)$ then $|S_{e_{ij}} - S_{e_{ij+1}}| = \frac{1}{3^{i+1}}$ Done!

$wt(E) = \Omega(wt(MST(S)) \log n)(proof)$

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$$n = 3^k - 1 \Rightarrow k = \log(n+1)$$

$$wt(E) = \frac{1}{3}(\log(n+1)) = \Omega(logn)$$

$wt(E) = \Omega(wt(MST(S)) \log n)(proof)$

- min element of S is $S = \frac{1}{3^k}$
- max element of S is $S = 1 - \frac{1}{3^k}$

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- min element of S is $S = \frac{1}{3^k}$
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- Since MST is consist of the sorted elements sequence:

$$wt(MST(S)) = 1 - \frac{2}{3^k} < 1$$

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- By applying these bounds :

$$wt(E) \times wt(MST(S)) < wt(E)$$

$$\Omega(\log n) \times wt(MST(S)) < wt(E) \times wt(MST(S)) < wt(E)$$

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$$wt(E) = \Omega(wt(MST(S)))\log n$$

Definitions

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- If d is large then $C_{dw} = 1 + \frac{2^{2d+1}}{C_d W_d}$
- $\Gamma(x+1) = x!$
- $\Gamma(x) = (x-1)\Gamma(x-1)$
- $\Gamma(1/2) = \sqrt{\pi}/2$

An Upper Bound(cont.)

Theorem 6.3.1:

- Let S be the set of n points in the d-dimensional unit cube $[0,1]^d$
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Theorem 6.3.1:

- Let S be the set of n points in the d-dimensional unit cube $[0, 1]^d$
- And E be the set of directed edges that satisfies the w-gap property
- for $d \geq 2$ and we have:

$$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$$

- $C_{dw} = 1 + \frac{2^{2d+1}}{C_d W_d}$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $|S| = n$ so $|E|$ is at most n
- we partition E according length
- $E = \begin{cases} E_l = (p,q) \in E : |pq| > n^{-1/d} \\ E_s = (p,q) \in E : |pq| \leq n^{-1/d} \end{cases}$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- In case of E_s:

$$wt(E_s) \leq |E_s| \times n^{\frac{-1}{d}} \leq n \times n^{\frac{-1}{d}} = n^{1-\frac{1}{d}}$$

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- In case of E_l:

$$I_j = (\frac{2^j}{n^{\frac{1}{d}}}, \frac{2^{j+1}}{n^{\frac{1}{d}}})$$

$$l = \frac{2^j}{n^{\frac{1}{d}}}$$

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$$l = \frac{2^j}{n^{\frac{1}{d}}}$$

- $F_j = \{(p, q) \in E_l : |pq| \in I_j\}$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $if(p,q) \in E_l \rightarrow n^{\frac{-1}{d}} < |pq|$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $if(p, q) \in E_l \rightarrow n^{\frac{-1}{d}} < |pq|$
- So $j \in [0, [\log \sqrt{d} n^{d-1}]]$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $|F_j| = k$
- $F_j = \{(p_i, q_i) : 1 \leq i \leq k\}$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $|F_j| = k$
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- $$L = \frac{2^j}{n^{\frac{1}{d}}} \to L < |p_i q_i| \leq 2L$$

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- Edges of E satisfy the w-gap property
- $\Rightarrow |p_i p_i'| > w.\min(|p_i q_i|, |p_i' q_i'|) > wL$

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- Edges of E satisfy the w-gap property
- $\Rightarrow |p_i p_i'| > w.\min(|p_i q_i|, |p_i' q_i'|) > wL$
- If we draw a d-dimensional ball B_i of radius $\frac{wL}{2}$ around each point $p_i, (1 \leq i \leq k)$, then these balls are pairwise disjoint.

$wt(E) \leq C_{dw} n^{1-\frac{1}{d}}$

- $\frac{WL}{2} \leq \frac{w\sqrt{d}}{2} \leq 1$

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- so at least a fraction of $(\frac{1}{2})^d$ of each ball B_i is contained in the unit cube.
- volume of d-dimensional ball of radius $R = C_d R^d$
- $c_d = \frac{\Pi^{\frac{d}{2}}}{\Gamma(d/2+1)}$

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- $k(\frac{1}{2})^d C_d(\frac{WL}{2})^d \leq 1$
- $\Rightarrow k \leq \frac{2^{2d}}{C_d W^d L^d}$

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$$\forall (p_i, q_i) \in F_j \qquad |p_i q_i| \leq 2L \Rightarrow wt(F_j) \leq 2LK \leq 2L \frac{2^{2d}}{C_d W^d L^d}$$

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$$wt(F_j) \leq \frac{2^{2d+1}}{C_d W^d 2^{j(d-1)}} \times n^{1-\frac{1}{d}}$$

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- $wt(E) = wt(E_s) + wt(E_l) \geq n^{1-\frac{1}{d}} (1 + \frac{2^{2d+2}}{C_d W^d})$

Reminder Lemma 4.1.4)

- Let $k \geq 8$, $\theta = \frac{2\pi}{k}$, and C be the cone such that $q \in C_p$

6.4)A useful geometric Lemma

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  $|pr| \leq |pq| \cos \theta$

  $|rp| \leq |pq| - (\cos \theta - \sin \theta)|pr|$

Lemma 6.4.1

Let $0 < \theta < \pi/4, 0 \leq w < (\cos\theta - \sin\theta)/2$,
$t \geq 1/(\cos\theta - \sin\theta - 2w)$ and $p, q, r, s \in R^d$ and:

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6.4)A useful geometric Lemma

$$Then = \begin{cases} |pr| < |pq| \\ |sq| < |pq| \\ t|pr| + |rs| + t|sq| \leq t|pq| \end{cases}$$

Lemma 6.4.1(proof)

$$|rs| \leq |pq| / \cos\theta \to |rs| < \sqrt{2}|pq| \qquad (0 < \theta < \pi/4)$$

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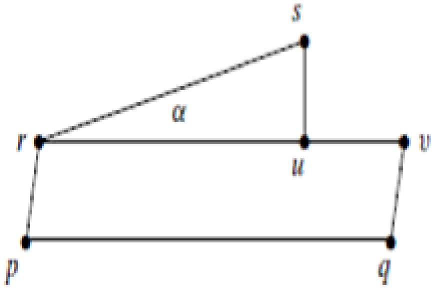

case1:

$|ru| \leq |rv|$

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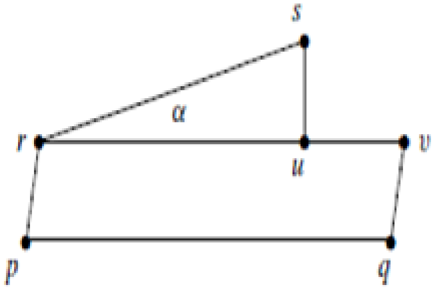

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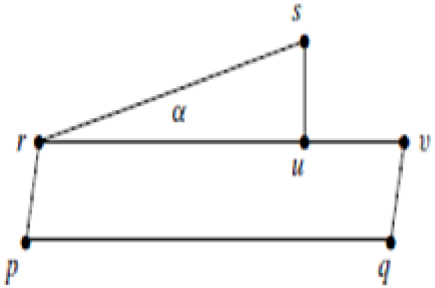

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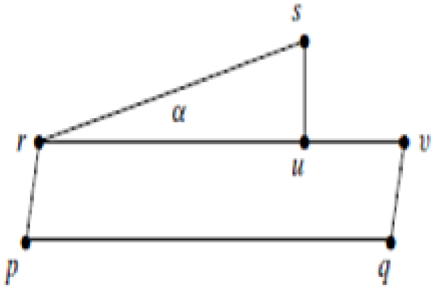

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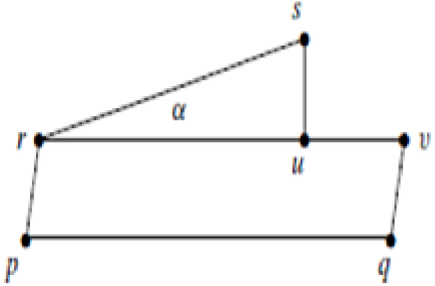

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$$tw|rs| + |rs| + t|pq| - t|rs|(\cos\theta - \sin\theta - w) \leq$$

$$= (1 - t(\cos\theta - \sin\theta - 2w))|rs| + t|pq|$$

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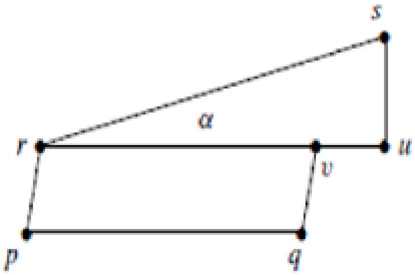

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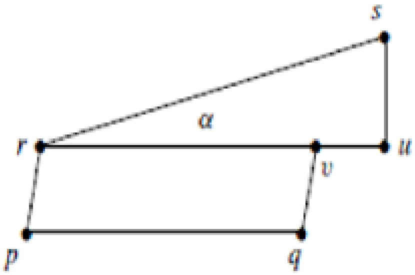

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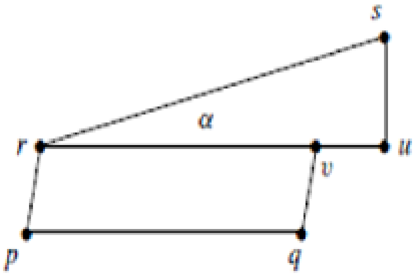

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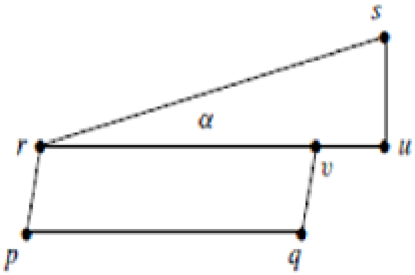

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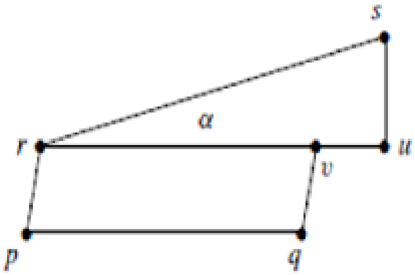

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&= (1 + t(\sin\theta + 2w))|rs|
\end{aligned}
$$

Lemma 6.4.1(proof)

$$t|pq| - t|pq| + (1 + t(\sin\theta + 2w))|rs|$$

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Worst case analysis of the 2-OPT algorithm for the TSP

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$TSP(S) \rightarrow T_0$

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Compute the approximation factor by Gap-theorem

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Compute the approximation factor by Gap-theorem

$\frac{wt(T_0)}{wt(TSP(S))} = ?$

We will show by gap property and prove
$wt(T_0)/wt(TSP(S)) = O(\log n)$

T_0:A tour along the points of s that is 2-optimal

Worst case analysis of the 2-OPT algorithm for the TSP

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$0 < \theta < \pi/4$, $0 < w < (\cos\theta - \sin\theta)/2$

Lemma 6.5.1

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then $|pr| > w.\min(|pq|, |rs|)$
((p, q) and (r, s) satisfy the w-gap property)

we prove the lemma by contradiction

assume that $|rs| \geq |pq|$ so $w. \min(|pq|, |rs|)$ is $|rs|$

Now assume by contradiction that $|pr| \geq w|rs|$

let $t = 1/(\cos\theta - \sin\theta - 2w)$

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$\Rightarrow t|pr| + |rs| + t|sq| \leq t|pr| + t|sq| - t|rs|$

$\Rightarrow (1 + t)|rs| \leq 0$

this is a contradiction, because $r \neq s$

so the edges satisfy the w-gap property

worst-case analysis of the 2-OPT TSP

- partition T_0 into $O(1/\theta^d - 1)$ subsets s.t. ant two edges make angle of at most θ (theorem 5.3.3 makes it possible)

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we can partition E into $O(1/\theta^d - 1)$ subsets in time $O(1/\theta^{d-1} + |E| \log 1/\theta)$

for each pair in the same subset $(pq, rs) \leq \theta$

Worst case analysis of the 2-OPT TSP

- (By 6.5.1 and 6.1.2) If a set of edges satisfies the w-gap property , then the weight of these edges is less than $(1 + \frac{2}{w})wt(MST(S)) \log n$

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- $wt(MST(S)) \leq wt(TSP(S))$
- $wt(T_0) = O(\frac{1}{w\theta^{d-1}} wt(TSP(S)) \log n)$

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- The worst-case approximation ratio of the 2-OPT algorithm is $O(\log n)$
- The ratio will be bigger than $\frac{c \log n}{\log \log n}$ for so many 'n'

Open problem

for all $s \in R^d$ and all 2-opt tours T_0 find the biggest value for $\frac{wt(T_0)}{wt(TSP(S))}$


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