I/O-Algorithms

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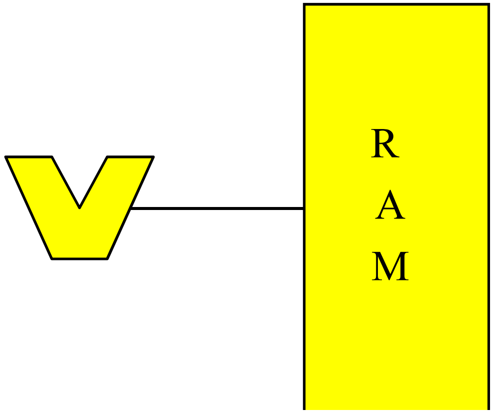
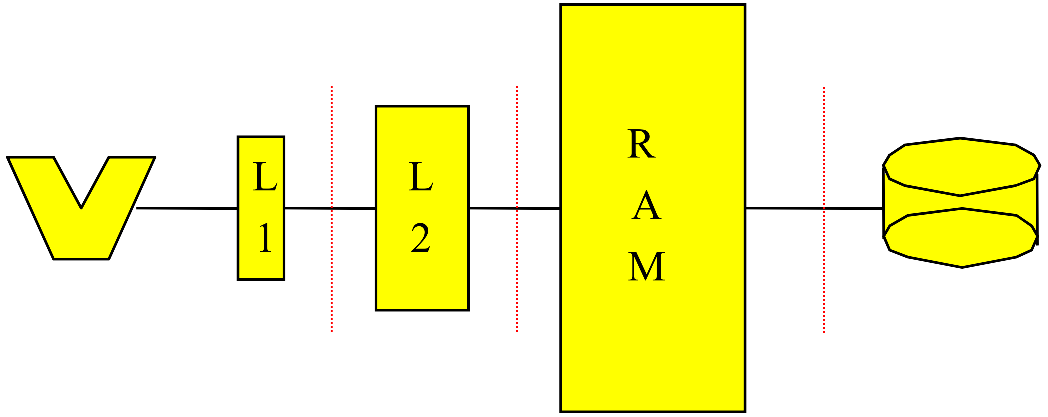
Random Access Machine Model



- Standard theoretical model of computation:
  - Infinite memory
  - Uniform access cost

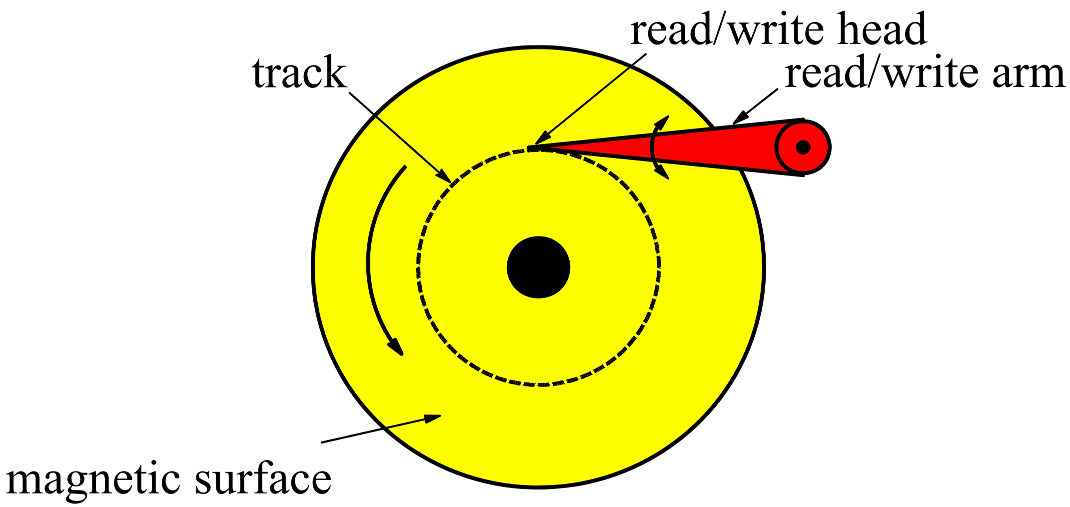
Hierarchical Memory



- Modern machines have complicated memory hierarchy
  - Levels get larger and slower further away from CPU
  - Large access time amortized using block transfer between levels
- Bottleneck often transfers between largest memory levels in use

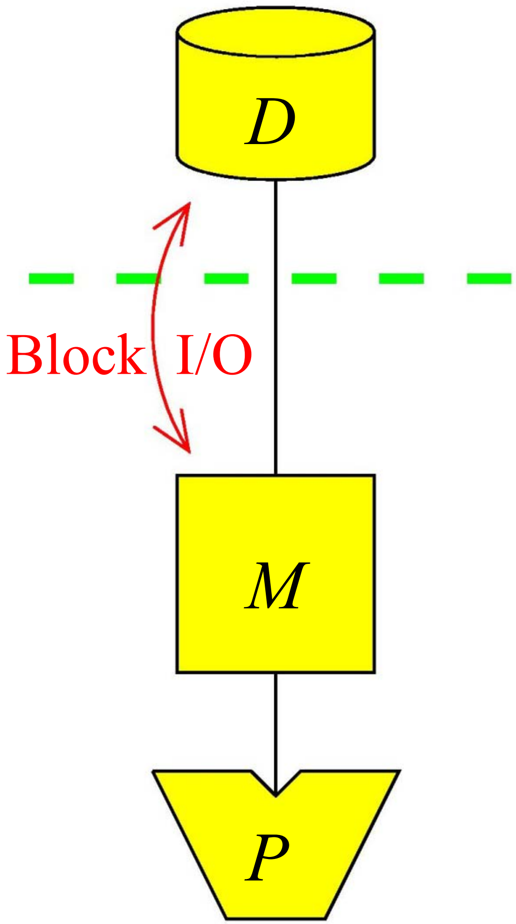
I/O-Bottleneck

- I/O is often bottleneck when handling massive datasets
  - Disk access is 10^6 times slower than main memory access
  - Large transfer block size (typically 8-16 Kbytes)



- Important to obtain "locality of reference"
  - Need to store and access data to take advantage of blocks

I/O-Model



- Parameters

  N = # elements in problem instance

  B = # elements that fits in disk block

  M = # elements that fits in main memory

  T = # output size in searching problem

- We often assume that $M>B^2$

- I/O: Movement of block between memory and disk

Fundamental Bounds

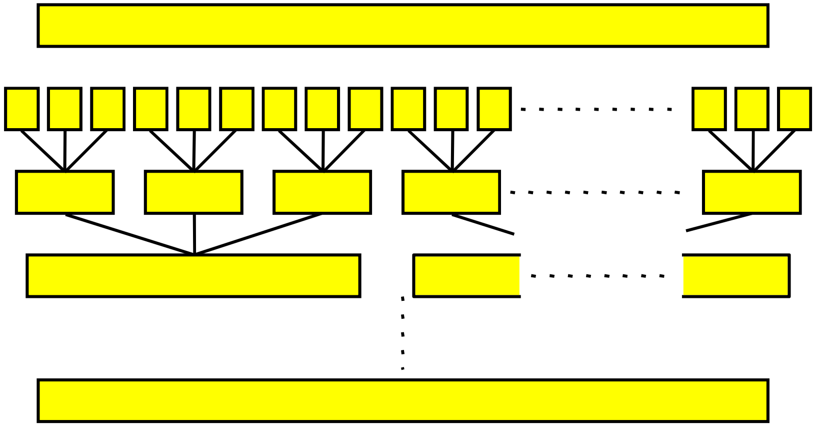
	Internal	External
• Scanning:	N	$\frac{N}{B}$
• Sorting:	$N \log N$	$\frac{N}{B}\log_{M/B}\frac{N}{B}$
• Permuting	N	$\min\{N, \frac{N}{B}\log_{M/B}\frac{N}{B}\}$
• Searching:	$\log_2 N$	$\log_B N$

- Note:
  - Linear I/O: $O(N/B)$
  - Permuting not linear
  - Permuting and sorting bounds are equal in all practical cases
  - B factor VERY important: $\frac{N}{B} < \frac{N}{B}\log_{M/B}\frac{N}{B} << N$
  - Cannot sort optimally with search tree

Merge Sort

- Merge sort:
  - Create N/M memory sized sorted runs
  - Merge runs together M/B at a time

$\Rightarrow O(\log_{M/B} \frac{N}{M})$ phases using $O(N/B)$ I/Os each



- Distribution sort similar (but harder – partition elements)

Permuting Lower Bound

Permuting N elements according to a given permutation takes $\Omega(\min\{N, \frac{N}{B}\log_{M/B}\frac{N}{B}\})$ I/Os in "indivisibility" model

- Indivisibility model: Move of elements only allowed operation
- Note:
  - We can allow copies (and destruction of elements)
  - Bound also a lower bound on sorting

- Proof:
  - View memory and disk as array of N tracks of B elements
  - Assume all I/Os track aligned (assumption can be removed)

Permuting Lower Bound

- Array contains permutation of N elements at all times
- We will count how many permutations can be reached (produced) with t I/Os
- *Input:*
  * Choose track: N possibilities
  * Rearrange $\leq B$ element in track and place among $\leq M$-B elements in memory:
    - $\leq B! \cdot \binom{M}{B}$ possibilities if “fresh” track
    - $\leq \binom{M}{B}$ otherwise

  $\Rightarrow$ at most $(N \cdot \binom{M}{B})^t \cdot (B!)^{N/B}$ permutations after t inputs
- *Output:*
  * Choose track: N possibilities

Permuting Lower Bound

– Permutation algorithm needs to be able to produce *N!* permutations

$$(N \cdot \binom{M}{B})^t \cdot (B!)^{N/B} \geq N! \qquad \Downarrow$$

$$\frac{N}{B}\log(B!) + t(\log N + \log\binom{M}{B}) \geq \log(N!) \qquad \Downarrow$$

$$N \log B + t(\log N + B \log \frac{M}{B}) \geq N \log N \qquad \Downarrow$$

$$t \geq \frac{N \log \frac{N}{B}}{\log N + B \log \frac{M}{B}}$$

(using Stirlings formula $\log x! \approx x \log x$ and $\log\binom{M}{B} \approx B \log \frac{M}{B}$)

– If $\log N \leq B \log \frac{M}{B}$ we have $t \geq \frac{N \log \frac{N}{B}}{2B \log \frac{M}{B}} = \Omega\left(\frac{N}{B} \log_{M/B} \frac{N}{B}\right)$

– If $\log N > B \log \frac{M}{B}$ we have $B << \sqrt{N}$ and thus

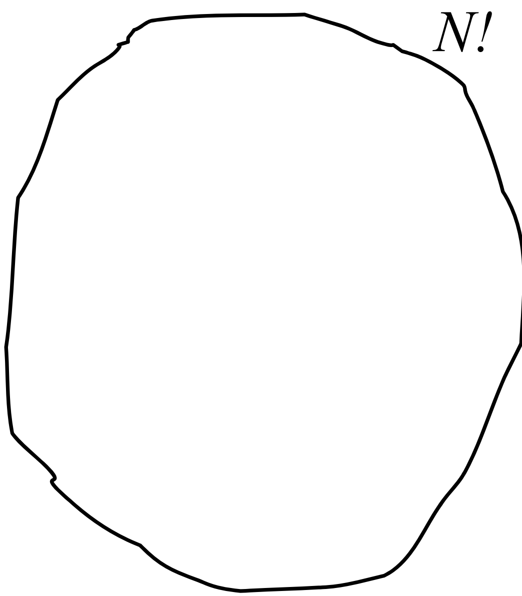
$$t \geq \frac{N \log \frac{N}{B}}{2 \log N} = \frac{1}{2}\left(N - N \frac{\log B}{\log N}\right) \geq \frac{1}{2}\left(N - \frac{1}{2} N\right) = \Omega(N)$$

$$\Rightarrow \quad t = \Omega(\min\{N, \frac{N}{B} \log_{M/B} \frac{N}{B}\})$$

Sorting lower bound

Sorting N elements takes $\Omega(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os in comparison model

- Proof:
  - Initially N elements stored in N/B first blocks on disk
  - Initially all $N!$ possible orderings consistent with our knowledge
  - After t I/Os?



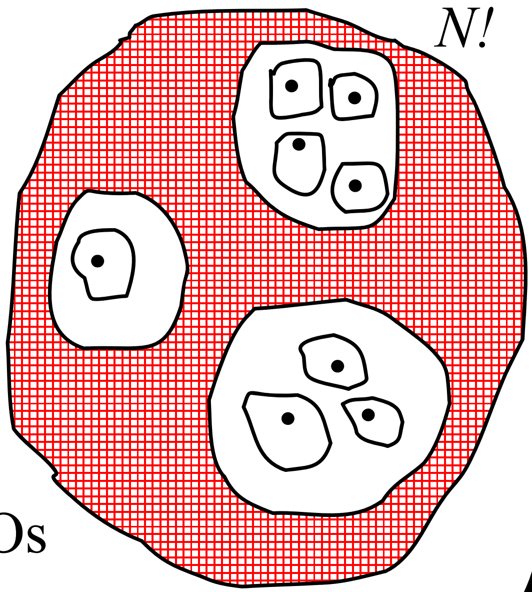
Sorting lower bound

- Consider one input assuming:
  - S consistent orderings before input
  - Compute total order of elements in memory
  - Adversary choose ”worst” outcome of comparisons done
- $\leq \binom{M}{B} \cdot B!$ possible orderings of M-B ”old” and B new elements in memory
- Adversary can choose outcome such that still $\geq S/(\binom{M}{B} \cdot B!)$ consistent orderings
- Only get $B!$ term N/B times

$\Downarrow$

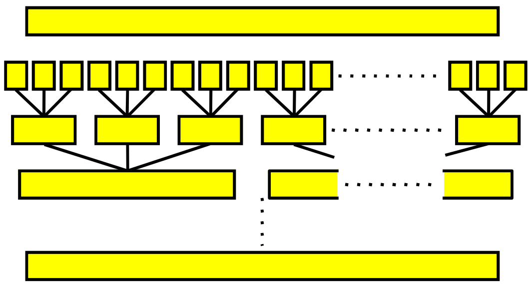
$\geq N!/(\binom{M}{B}^t \cdot (B!)^{N/B})$ consistent orderings after t I/Os

$\Rightarrow N!/(\binom{M}{B}^t \cdot (B!)^{N/B}) = 1 \;\Rightarrow\; t = \Omega(\frac{N}{B} \log_{M/B} \frac{N}{B})$



Summary/Conclusion: Sorting

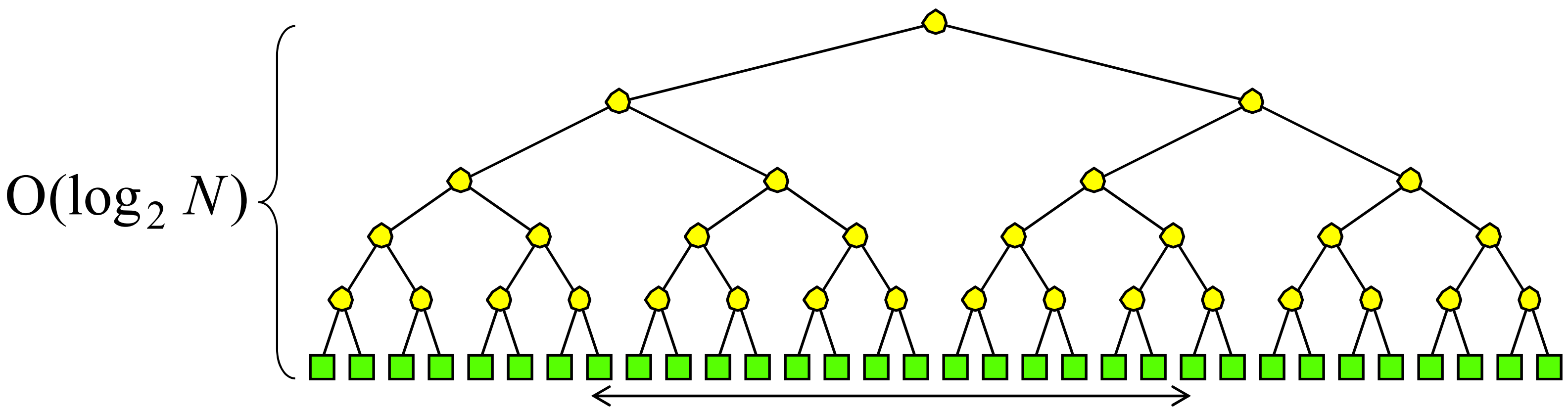
- External merge or distribution sort takes $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
  - Merge-sort based on M/B-way merging
  - Distribution sort based on $\sqrt{M/B}$-way distribution and partition elements finding



- Optimal in comparison model
- Can prove $\Omega(\min\{N, \frac{N}{B}\log_{M/B}\frac{N}{B}\})$ lower bound in stronger model
  - Holds even for permuting

External Search Trees

- Binary search tree:
  - Standard method for search among N elements
  - We assume elements in leaves

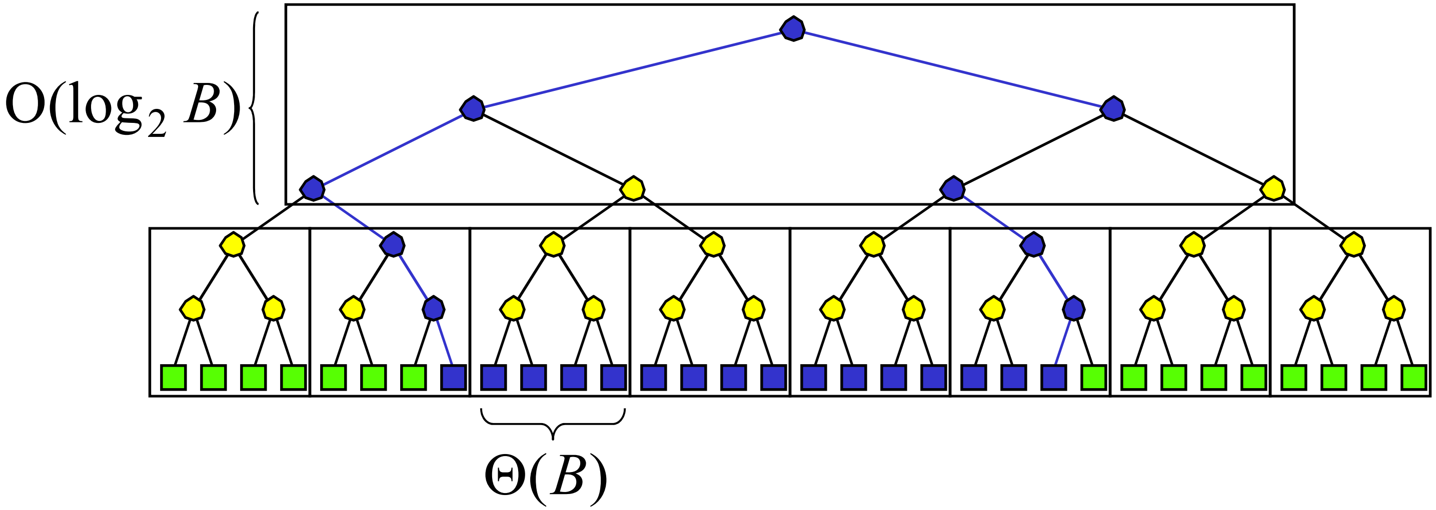


  - Search traces at least one root-leaf path
  - If nodes stored arbitrarily on disk

    $\Rightarrow$ Search in $O(\log_2 N)$ I/Os

    $\Rightarrow$ Rangesearch in $O(\log_2 N + T)$ I/Os

External Search Trees



- BFS blocking:
  - Block height $O(\log_2 N)/O(\log_2 B) = O(\log_B N)$
  - Output elements blocked

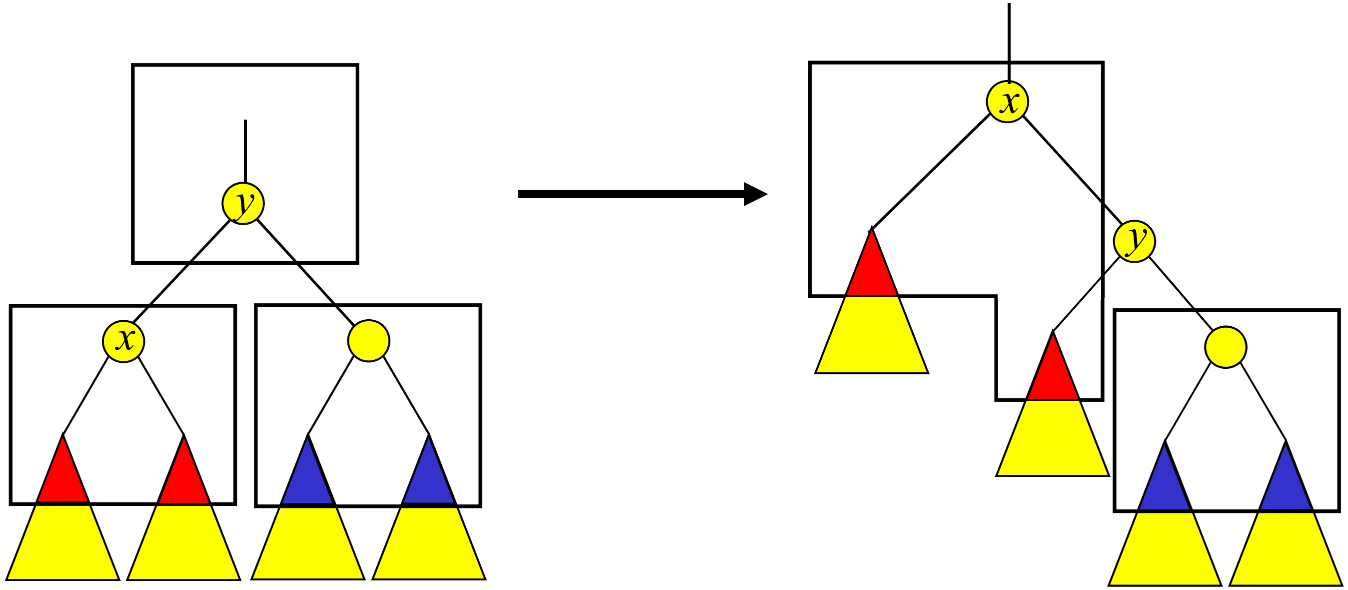
$\Downarrow$

Rangesearch in $O(\log_B N + T/B)$ I/Os

- Optimal: $O(N/B)$ space and $O(\log_B N + T/B)$ query

External Search Trees

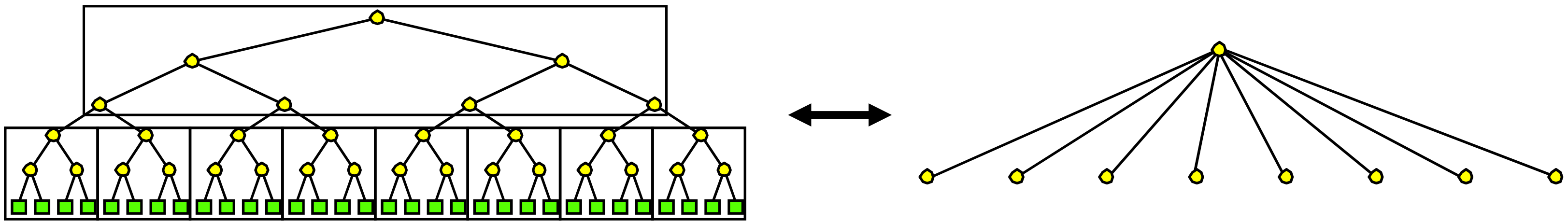
- Maintaining BFS blocking during updates?
  - Balance normally maintained in search trees using rotations



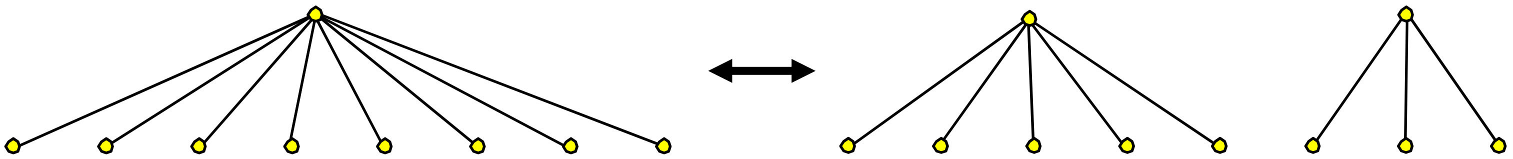
- Seems very difficult to maintain BFS blocking during rotation
  - Also need to make sure output (leaves) is blocked!

B-trees

- BFS-blocking naturally corresponds to tree with fan-out $\Theta(B)$

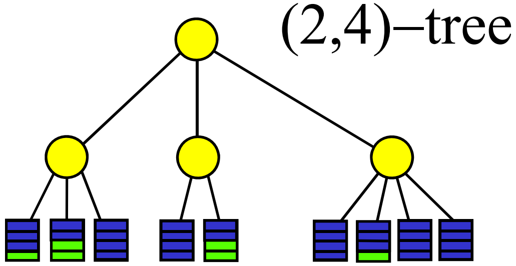


- B-trees balanced by allowing node degree to vary
  – Rebalancing performed by splitting and merging nodes



(a,b)-tree

- T is an (a,b)-tree ($a \geq 2$ and $b \geq 2a-1$)
  - All leaves on the same level and contain between a and b elements
  - Except for the root, all nodes have degree between a and b
  - Root has degree between 2 and b



- (a,b)-tree uses linear space and has height $O(\log_a N)$

$\Downarrow$

Choosing $a,b = \Theta(B)$ each node/leaf stored in one disk block

$\Downarrow$

$O(N/B)$ space and $O(\log_B N + T/B)$ query

(a,b)-Tree Insert

- Insert:

Search and insert element in leaf v

DO v has $b+1$ elements/children

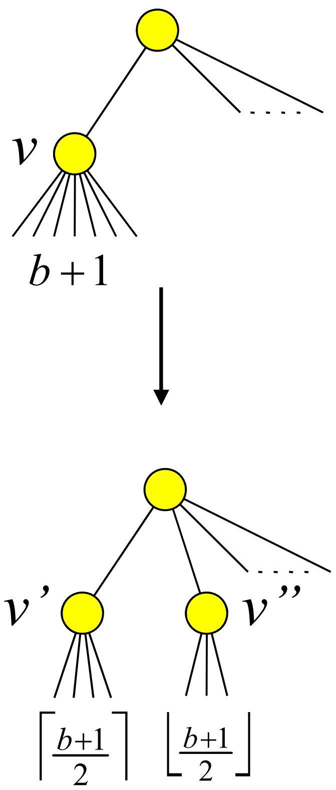
  Split v:

    make nodes v' and v'' with $\left\lceil \frac{b+1}{2} \right\rceil \leq b$ and $\left\lfloor \frac{b+1}{2} \right\rfloor \geq a$ elements

    insert element (ref) in *parent(v)*

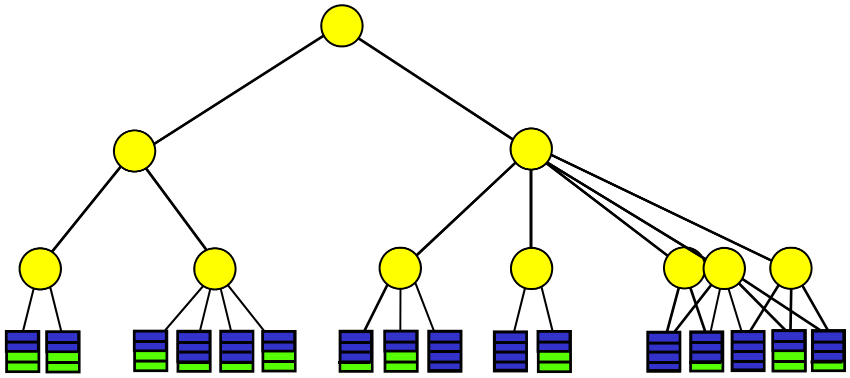
    (make new root if necessary)

  v=parent(v)



- Insert touch $O(\log_a N)$ nodes

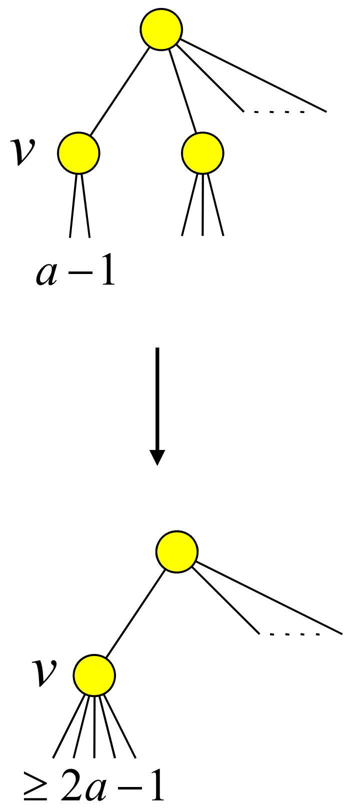
(*2*,*4*)-Tree Insert



(a,b)-Tree Delete

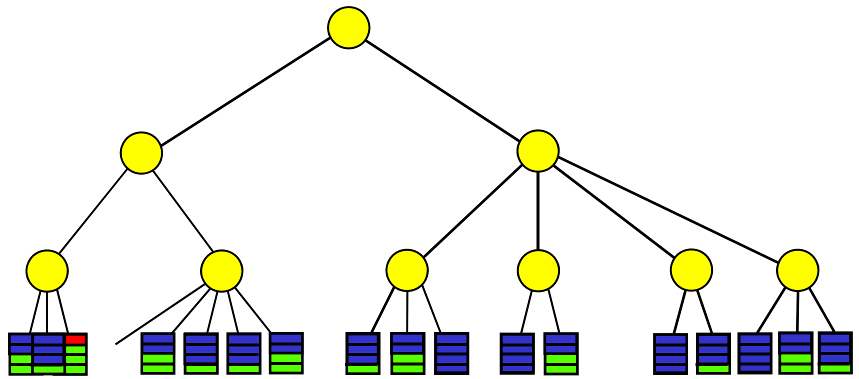
- Delete:

Search and delete element from leaf v
DO v has a-1 elements/children
  Fuse v with sibling v':
    move children of v' to v
    delete element (ref) from $parent(v)$
    (delete root if necessary)
  If v has $>b$ (and $\leq a+b-1<2b$) children split v
$v=parent(v)$



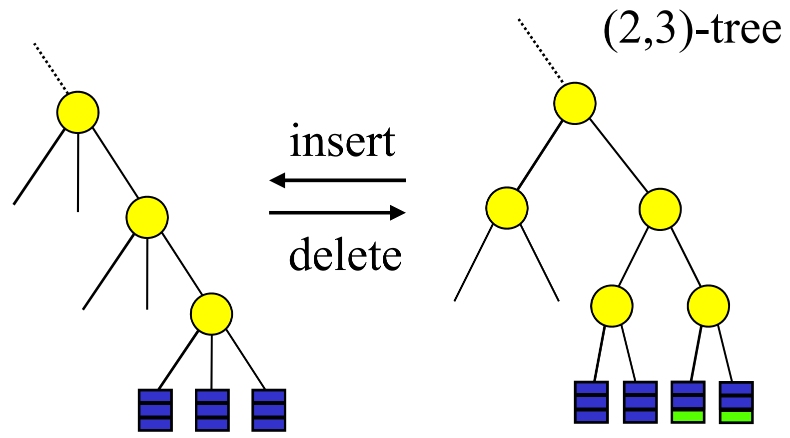
- Delete touch $O(\log_a N)$ nodes

(2,4)-Tree Delete



(*a*,*b*)-Tree

- (a,b)-tree properties:
  - If $b=2a-1$ every update can cause many rebalancing operations



  - If $b \geq 2a$ update only cause $O(1)$ rebalancing operations amortized
  - If $b>2a$ only $O(\frac{1}{b/2-a}) = O(1/a)$ rebalancing operations amortized
    - * Both somewhat hard to show
  - If $b=4a$ easy to show that update causes $O(\frac{1}{a}\log_a N)$ rebalance operations amortized
    - * After split during insert a leaf contains $\simeq 4a/2=2a$ elements
    - * After fuse during delete a leaf contains between $\simeq 2a$ and $\simeq 5a$ elements (split if more than $3a \Rightarrow$ between $3/2a$ and $5/2a$)

Summary/Conclusion: B-tree

- B-trees: (a,b)-trees with $a,b = \Theta(B)$
  - $O(N/B)$ space
  - $O(\log_B N + T/B)$ query
  - $O(\log_B N)$ update
- B-trees with elements in the leaves sometimes called B^+-tree
- Construction in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/Os
  - Sort elements and construct leaves
  - Build tree level-by-level bottom-up

Summary/Conclusion: B-tree

- B-tree with branching parameter b and leaf parameter k ($b,k \geq 8$)
  - All leaves on same level and contain between $1/4k$ and k elements
  - Except for the root, all nodes have degree between $1/4b$ and b
  - Root has degree between 2 and b

- B-tree with leaf parameter $k = \Omega(B)$
  - $O(N/B)$ space
  - Height $O(\log_b \frac{N}{B})$
  - $O(1/k)$ amortized leaf rebalance operations
  - $O(\frac{1}{b \cdot k} \log_b \frac{N}{B})$ amortized internal node rebalance operations
- B-tree with branching parameter B^c, $0 < c \leq 1$, and leaf parameter B
  - Space $O(N/B)$, updates $O(\log_B N)$, queries $O(\log_B N + T/B)$

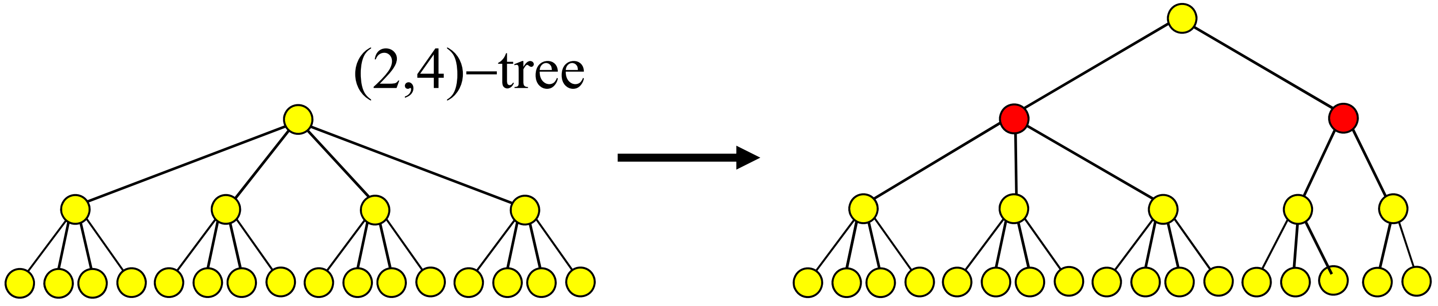
Secondary Structures

- When secondary structures used, a rebalance on v often requires $O(w(v))$ I/Os ($w(v)$ is *weight* of v)
  - If $\Omega(w(v))$ inserts have to be made below v between operations

    $\Rightarrow O(1)$ amortized split bound

    $\Rightarrow O(\log_B N)$ amortized insert bound

- Nodes in standard B-tree do not have this property



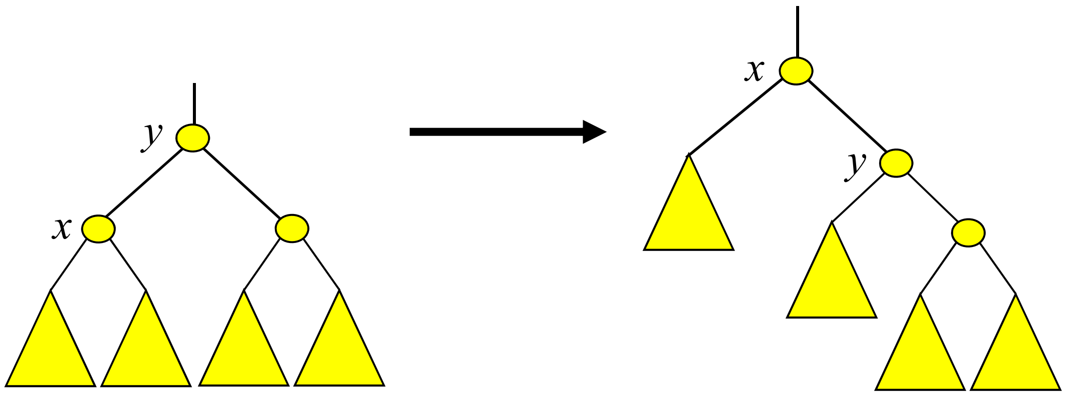
BB[α]-tree

- In internal memory BB[α]-trees have the desired property
- Defined using weight-constraint
  - Ratio between weight of left child and weight of right child of a node v is between α and 1-α (α<1)

$\Downarrow$

Height $O(\log N)$

- If $2/11 < \alpha < 1 - \frac{1}{2}\sqrt{2}$ rebalancing can be performed using rotations

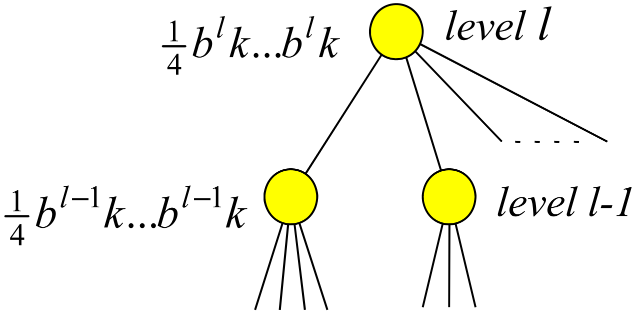


- Seems hard to implement BB[α]-trees I/O-efficiently

Weight-balanced B-tree

- Idea: Combination of B-tree and BB[α]-tree
  - Weight constraint on nodes instead of degree constraint
  - Rebalancing performed using split/fuse as in B-tree

- Weight-balanced B-tree with parameters b and k (b>8, $k\geq 8$)
  - All leaves on same level and contain between $k/4$ and k elements
  - Internal node v at level l has $w(v) < b^l k$
  - Except for the root, internal node v at level l has $w(v) > \frac{1}{4} b^l k$
  - The root has more than one child

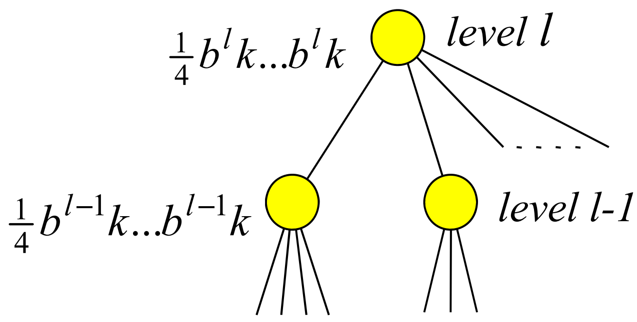


Weight-balanced B-tree

- Every internal node has degree between
  $\frac{1}{4}b^l k / b^{l-1}k = \frac{1}{4}b$ and $b^l k / \frac{1}{4}b^{l-1}k = 4b$

  $\Downarrow$

  Height $O(\log_b \frac{N}{k})$



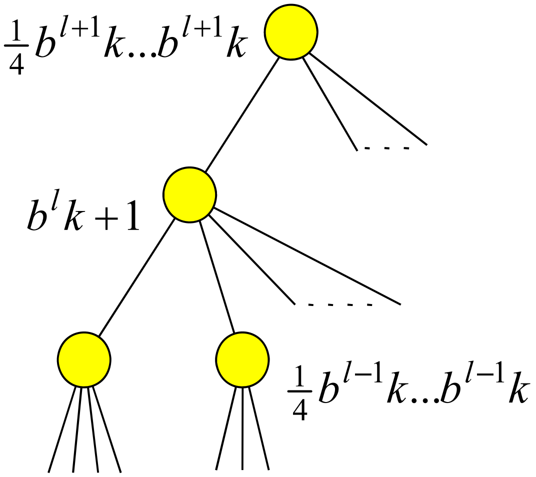
- External memory:
  - Choose $4b=B$ (or even B^c for $0 < c \leq 1$)
  - $k=B$

  $\Downarrow$

  $O(N/B)$ space, $O(\log_B N + T/B)$ query

Weight-balanced B-tree Insert

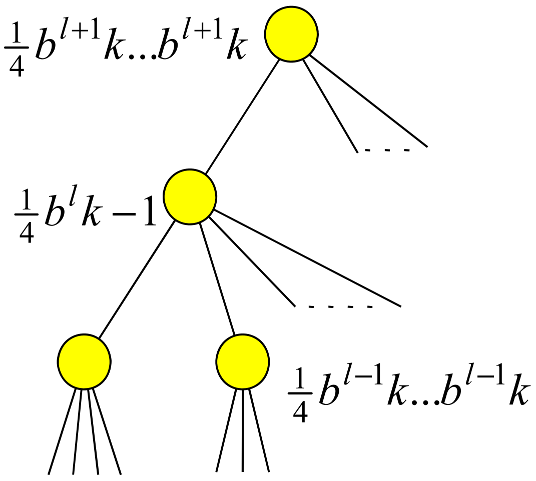
- Search for relevant leaf u and insert new element
- Traverse path from u to root:
  - If level l node v now has $w(v)=b^l k+1$ then split into nodes v' and v'' with $w(v') \geq \lfloor \frac{1}{2}(b^l k+1) \rfloor - b^{l-1}k$ and $w(v'') \leq \lceil \frac{1}{2}(b^l k+1) \rceil + b^{l-1}k$



- Algorithm correct since $\leq b^{l-1}k \leq \frac{1}{8}b^l k$ such that $w(v') \geq \frac{3}{8}b^l k$ and $w(v'') \leq \frac{5}{8}b^l k$
  - touch $O(\log_b \frac{N}{k})$ nodes
- Weight-balance property:
  - $\Omega(b^l k)$ updates below v' and v'' before next rebalance operation

Weight-balanced B-tree Delete

- Search for relevant leaf u and delete element
- Traverse path from u to root:
  - If level l node v now has $w(v)=\frac{1}{4}b^l k-1$ then fuse with sibling into node v' with $\frac{2}{4}b^l k-1 \le w(v') \le \frac{5}{4}b^l k-1$
  - If now $w(v') \ge \frac{7}{8}b^l k$ then split into nodes with weight $\ge \frac{7}{16}b^l k-1-b^{l-1}k \ge \frac{5}{16}b^l k-1$ and $\le \frac{5}{8}b^l k+b^{l-1}k \le \frac{6}{8}b^l k$



- Algorithm correct and touch $O(\log_b \frac{N}{k})$ nodes
- Weight-balance property:
  - $\Omega(b^l k)$ updates below v' and v'' before next rebalance operation

Summary/Conclusion: Weight-balanced B-tree

- Weight-balanced B-tree with branching parameter b and leaf parameter $k=\Omega(B)$
  - $O(N/B)$ space
  - Height $O(\log_b \frac{N}{k})$
  - $O(\log_b N)$ rebalancing operations after update
  - $\Omega(w(v))$ updates below v between consecutive operations on v

- Weight-balanced B-tree with branching parameter B^c and leaf parameter B
  - Updates in $O(\log_B N)$ and queries in $O(\log_B N + T/B)$ I/Os

- Construction bottom-up in $O(\frac{N}{B}\log_{M/B}\frac{N}{B})$ I/O

References

- **Lower bound on External Permuting/Sorting**

  Lecture notes by L. Arge.

- **External Memory Geometric Data Structures**

  Lecture notes by Lars Arge.

  - Section 1-3