Computing Convex Hull (in 2D)

1393-1

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Convex hull
Definition
Geometry of problem
1st algorithm
2nd algorithm
Proof of correctness
Other algorithms
Higher dimensions

Convex Set

Definition:

A subset S of the plane is called convex if and only if for any pair of points $p, q \in S$, the line segment $\overline{pq}$ is completely contained in S.

Convex Hull:

The convex hull $\mathcal{CH}(S)$ of a set S is the smallest convex set that contains S. To be more precise, it is the intersection of all convex sets that contain S.

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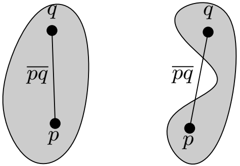

convex   not convex

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Observation:

It is the unique convex polygon whose vertices are points from P and that contains all points of P.

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Computing CH:

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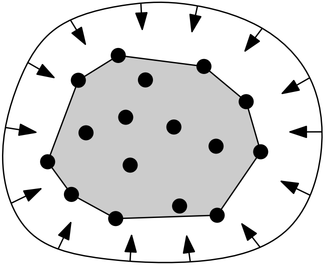

Computing CH:

Problem:

Given a set $P = \{p_1, p_2, \ldots, p_n\}$ of points in the plane, compute a list that contains those points from P that are the vertices of $\mathcal{CH}(P)$, listed in clockwise order.

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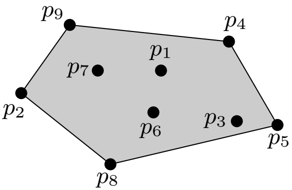
Input= set of points

$p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9$

Output= representation of the convex hull:

p_4, p_5, p_8, p_2, p_9



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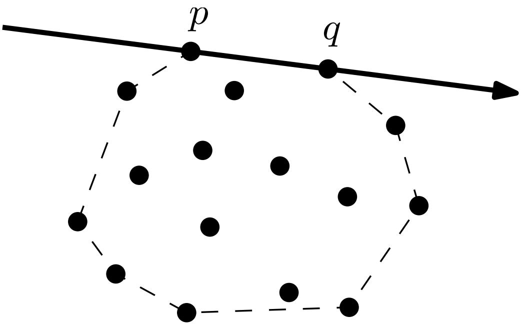
Geometry of the problem

Property:

If we direct the line through p and q such that $\mathcal{CH}(P)$ lies to the right, then all the points of P must lie to the right of this line. The reverse is also true: if all points of $P \setminus \{p, q\}$ lie to the right of the directed line through p and q, then pq is an edge of $\mathcal{CH}(P)$.



First algorithm

Algorithm SLOWCONVEXHULL(P)
Input: A set P of points in the plane.
Output: $\mathcal{L}$ =The vertices of $\mathcal{CH}(P)$ in clockwise order.

1. $E \leftarrow \emptyset$.
2. **for** all ordered pairs $(p, q) \in P \times P$ with $p \neq q$
3. $\quad$ $valid \leftarrow$ **true**
4. $\quad$ **for** all points $r \in P$ not equal to p or q
5. $\quad\quad$ **if** r lies to the left $\overrightarrow{pq}$
6. $\quad\quad\quad$ **then** $valid \leftarrow$ **false**.
7. $\quad$ **if** $valid$ then Add $\overrightarrow{pq}$ to E.
8. From the set E of edges construct vertices of $\mathcal{CH}(P)$, sorted in clockwise order.

Clarify:

1. How do we test whether a point lies to the left or to the right of a directed line? (See Exercise 1.4)
2. How can we construct $\mathcal{L}$ from E?

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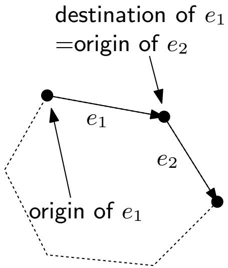
Computing $\mathcal{L}$:

- All edges are directed.
- remove an arbitrary edge e_1 from E. Put the origin of e_1 and the destination in $\mathcal{L}$.
- Find the edge e_2 in E whose origin is the destination of e_1, remove it from E, and append its destination to $\mathcal{L}$.
- Next, find the edge e_3 whose origin is the destination of e_2, remove it from E, and append its destination to $\mathcal{L}$.
- And so on.



Time Complexity: $\mathcal{O}(n^2)$

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From the set E of edges construct vertices of $\mathcal{CH}(P)$, sorted in clockwise order.

Running time: $\mathcal{O}(n^3) + \mathcal{O}(n^2) = \mathcal{O}(n^3)$.

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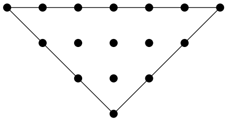
Degenerate case (or Degeneracy)

Degenerate Case:

A point r does not always lie to the right or to the left of the line through p and q, it can also happen that it **lies on** this line. What should we do then?



Solution:

A directed edge $\overrightarrow{pq}$ is an edge of $\mathcal{CH}(P)$ if and only if all other points $r \in P$ lie either strictly to the right of the directed line through p and q, or they lie on the open line segment $\overline{pq}$.

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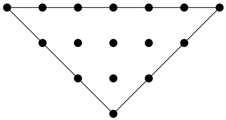

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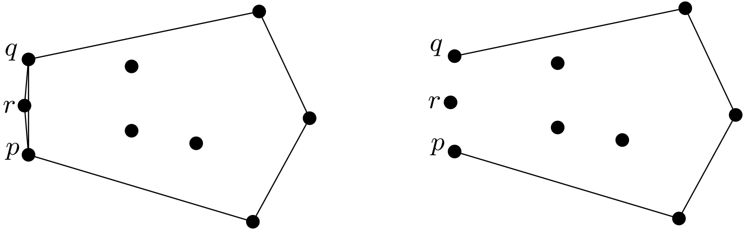
If the points are given in floating point coordinates and the computations are done using floating point arithmetic, then there will be rounding errors that may distort the outcome of tests.



This algorithm is not robust!

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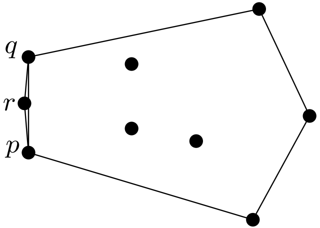

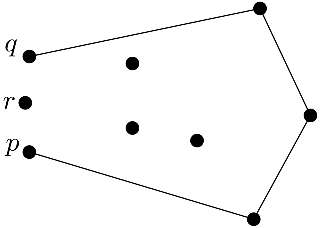

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2nd algorithm: incremental

Convex hull

Definition

Geometry of problem

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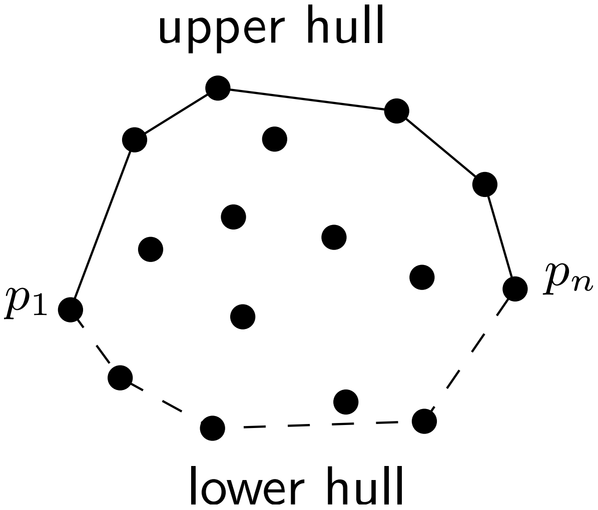
2nd algorithm: incremental



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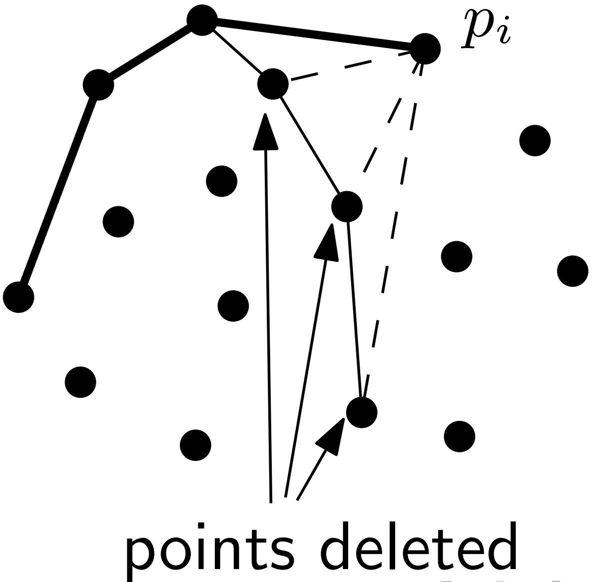

Algorithm CONVEXHULL(P)
Input: A set P of points in the plane.
Output: $\mathcal{L}$ =The vertices of $\mathcal{CH}(P)$ in clockwise order.

$p_1, \ldots, p_n$ = the points sorted by x-coordinate.
Add p_1 and p_2 to a list $\mathcal{L}_{upper}$, with p_1 as the first point.
for $i \leftarrow 3$ **to** n
    Append p_i to $\mathcal{L}_{upper}$.
    while $|\mathcal{L}_{upper}| > 2$ and the last 3 points in $\mathcal{L}_{upper}$ do not make a right turn
        Delete the middle of last 3 points from $\mathcal{L}_{upper}$.
Add p_n and p_{n-1} to a list $\mathcal{L}_{lower}$, with p_n as the first point.
for $i \leftarrow n-2$ downto 1
    Append p_i to $\mathcal{L}_{lower}$.
    while $|\mathcal{L}_{lower}| > 2$ and the last three points in $\mathcal{L}_{lower}$ do not make a right turn
        Delete the middle of last 3 points from $\mathcal{L}_{lower}$.
Remove the first and the last point from $\mathcal{L}_{lower}$.
Append $\mathcal{L}_{lower}$ to $\mathcal{L}_{upper}$, and call the resulting list $\mathcal{L}$.
return $\mathcal{L}$

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2nd algorithm: incremental

Special cases:

1. Two points have same x-coordinate.
2. Three points on a line

Solution:

Use the lexicographic order.

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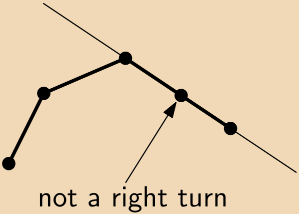

Robustness:

What does the algorithm do in the presence of rounding errors in the floating point arithmetic?

- When such errors occur, it can happen that a point is removed from the convex hull although it should be there, or that a point inside the real convex hull is not removed. But the structural integrity of the algorithm is unharmed: it will compute a closed polygonal chain.
- The only problem that can still occur is that, when three points lie very close together, a turn that is actually a sharp left turn can be interpreted as a right turn. This might result in a dent in the resulting polygon.

Proof of correctness:

Theorem: The convex hull of a set of n points in the plane can be computed in $\mathcal{O}(n \log n)$ time.

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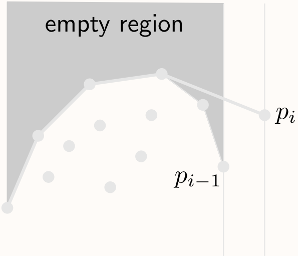

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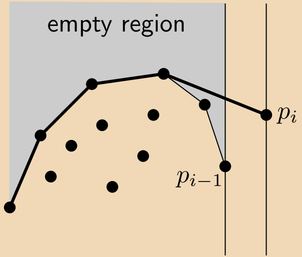

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Time Complexity:

- Sorting: $\mathcal{O}(n \log n)$.
- The for-loop is executed a linear number of times.
- For each execution of the for-loop the while-loop is executed at least once. For any extra execution a point is deleted from the current hull.
- So the time complexity for computing upper hull and lower hull is $\mathcal{O}(n)$.
- Total running time: $\mathcal{O}(n \log n)$.

Lower bound:

An $\Omega(n \log n)$ lower bound is known for the convex hull problem.

Convex hull

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Other algorithms:

Algorithm	Speed	Discovered By
Brute Force	$\mathcal{O}(n^4)$	[Anon, the dark ages]
Gift Wrapping	$\mathcal{O}(nh)$	[Chand & Kapur, 1970]
Graham Scan	$\mathcal{O}(n \log n)$	[Graham, 1972]
Jarvis March	$\mathcal{O}(nh)$	[Jarvis, 1973]
QuickHull	$\mathcal{O}(nh)$	[Eddy, 1977], [Bykat, 1978]
Divide-and-Conquer	$\mathcal{O}(n \log n)$	[Preparata & Hong, 1977]
Monotone Chain	$\mathcal{O}(n \log n)$	[Andrew, 1979]
Incremental	$\mathcal{O}(n \log n)$	[Kallay, 1984]
Marriage-before-Conquest	$\mathcal{O}(n \log h)$	[Kirkpatrick & Seidel, 1986]

n: number of points

h: number of points on the boundary of convex hull

Higher dimensions:

- The convex hull can be defined in any dimension.
- Convex hulls in 3-dimensional space can still be computed in $\mathcal{O}(n \log n)$ time (Chapter 11).
- For dimensions higher than 3, however, the complexity of the convex hull is no longer linear in the number of points.

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