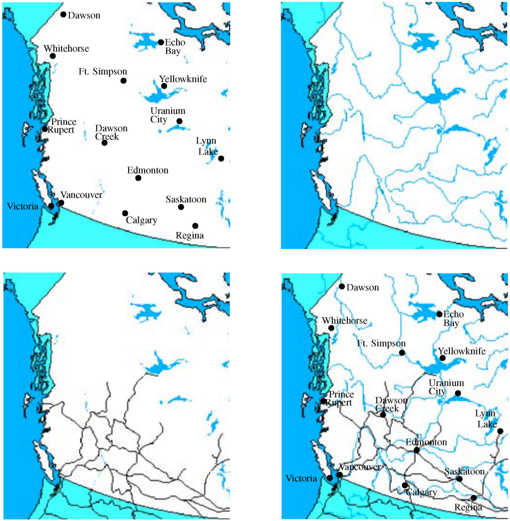
Line Segment Intersection

1393-1

Motivation

Thematic Map Overlay



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grizzly bear

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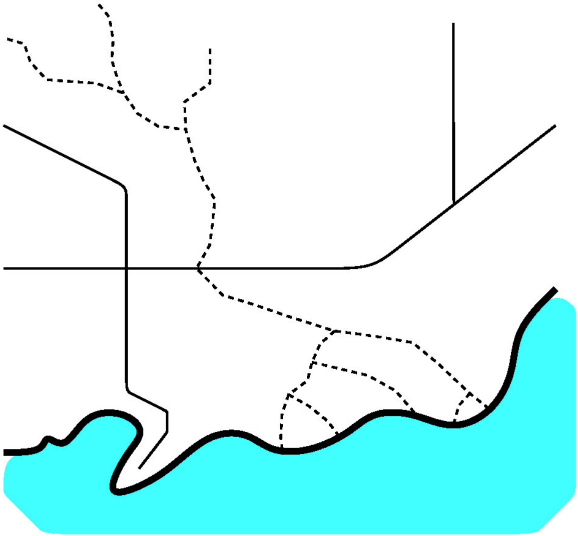

Problem

Line segment intersection problem:

Given two sets of line segments, compute all intersections between a segment from one set and a segment from the other.

$\star$ We consider the segments to be closed.

Simplified version:

Given a set S of n closed segments in the plane, report all intersection points among the segments in S.

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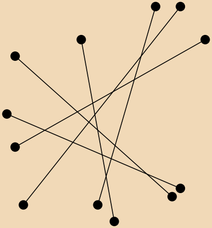

1st algorithm

- The brute-force algorithm clearly requires $\mathcal{O}(n^2)$ time.
- In a sense this is optimal: when each pair of segments intersects any algorithm must take $\Omega(n^2)$ time, because it has to report all intersections.

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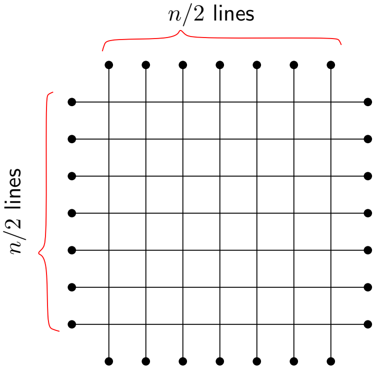

Output sensitive algorithm

Definition:

An algorithm whose running time depends not only on the number of segments in the input, but also on the number of intersection points.

In our case:

We want an algorithm that runs faster when the number of intersections is sub-quadratic.

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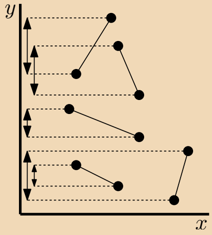
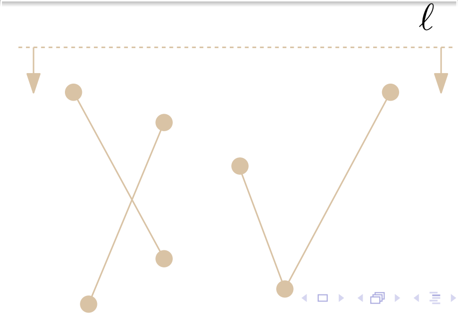
y-intervals

- Define the y-interval of a segment to be its orthogonal projection onto the y-axis.
- When the y-intervals of a pair of segments do not overlap then they cannot intersect.
- To find segments whose y-intervals overlap we use a **Plane sweep algorithm**.



Plane sweep algorithm

- We imagine sweeping a line ℓ downwards over the plane, starting from a position above all segments.
- While we sweep the imaginary line, we keep track of all segments intersecting it so that we can find the pairs we need.
- The **status** of the sweep line is the set of segments intersecting it.



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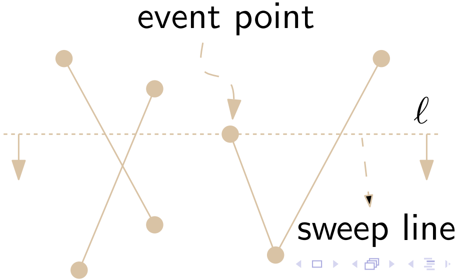

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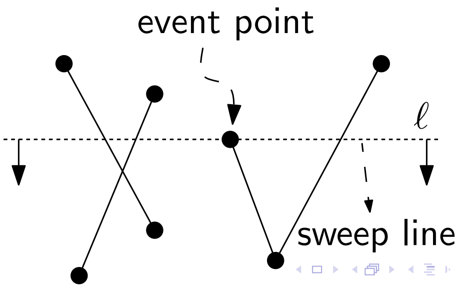

Plane sweep algorithm

when the sweep line reaches an event point:

- If the event point is the upper endpoint of a segment, then a new segment starts intersecting the sweep line and must be added to the status.
- If the event point is a lower endpoint, a segment stops intersecting the sweep line and must be deleted from the status.
- The algorithm tests pairs of segments for which there is a horizontal line that intersects both segments. (better than brute-force but still quadratic).

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New algorithm:

- Status: Sorted list of segments (from left to right as they intersect the sweep line).
- Test adjacent segments in the horizontal ordering for intersection.
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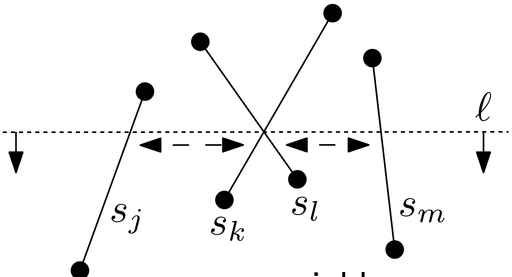

Do we still find all intersections?

Lemma 2.1 Let s_i and s_j be two non-horizontal segments whose interiors intersect in a single point p, and assume there is no third segment passing through p. Then there is an event point above p where s_i and s_j become adjacent and are tested for intersection.

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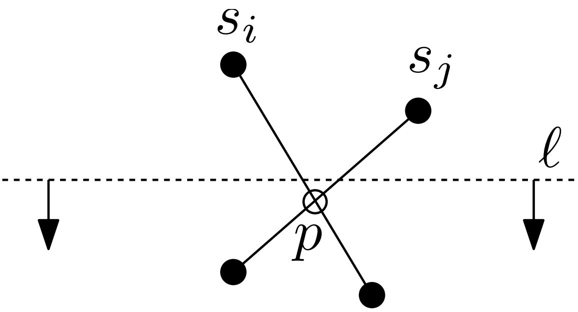

Handling event points:

The event point is the upper endpoint of a segment:

- Insert the new segment in the sorted list.
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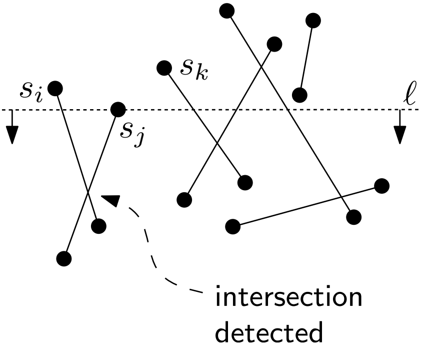

Plane sweep algorithm

Handling event points:

The event point is an intersection:

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- For each intersected segment, check for intersection between the segment and the new neighbor in the sorted list.

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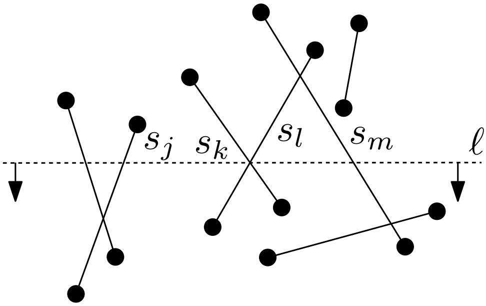

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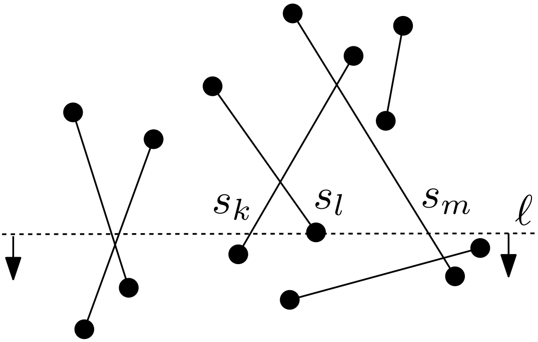

Plane sweep algorithm

A data structure for handling event:

We need an event queue $\mathcal{Q}$ such that:

1. find and removes the next event that will occur from $\mathcal{Q}$. If two event points have the same y-coordinate, then the one with smaller x-coordinate will be returned.
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Implementation of the event queue:

1. Define an order $\prec$ on event points: $p \prec q$ if and only if $p_y > q_y$ holds or $p_y = q_y$ and $p_x < q_x$ holds.
2. We store the event points in a balanced binary search tree, ordered according to $\prec$.
3. Fetching the next event and inserting an event and testing whether a given event is already present in $\mathcal{Q}$ take $\mathcal{O}(\log m)$ time, where m is the number of events in $\mathcal{Q}$.

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To maintain the sorted list of segments (status of the algorithm):

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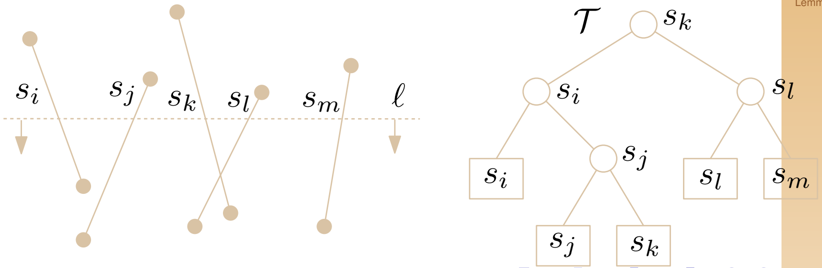

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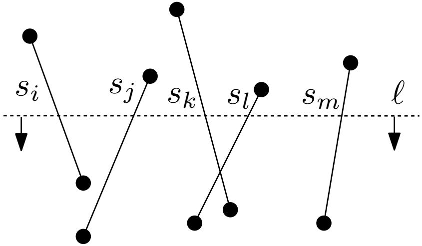

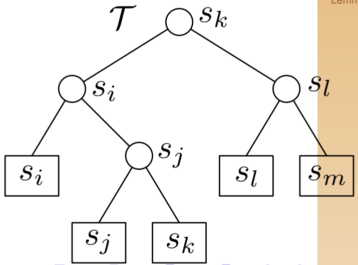

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Computational Geometry

Plane sweep algorithm

To search in $\mathcal{T}$ for the segment immediately to the left of some point p:

1. Traverse the tree until you meet a leaf.
2. This leaf, or the leaf immediately to the left of it, stores the segment we are searching for.
3. Therefore each update and neighbor search operation takes $\mathcal{O}(\log n)$ time.

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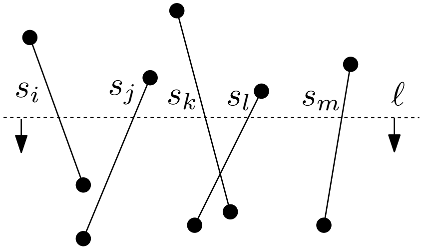
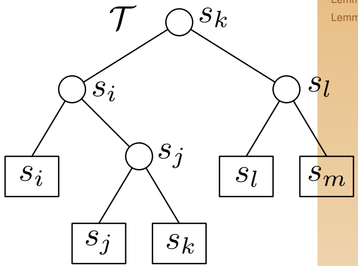




Algorithm FINDINTERSECTIONS(S)
Input: A set S of line segments in the plane.
Output: The set of intersection points among the segments in S, with for each intersection point the segments that contain it.

1. Initialize an empty event queue $\mathcal{Q}$. Next, insert the segment endpoints into $\mathcal{Q}$; when an upper endpoint is inserted, the corresponding segment should be stored with it.
2. Initialize an empty status structure $\mathcal{T}$.
3. **while** $\mathcal{Q}$ is not empty
4. Determine the next event point p in $\mathcal{Q}$ and delete it.
5. HANDLEEVENTPOINT(p)

Plane sweep algorithm

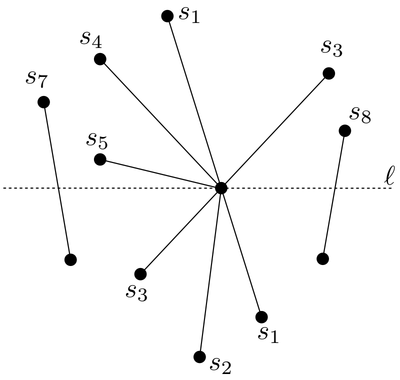
Algorithm HANDLEEVENTPOINT(p)

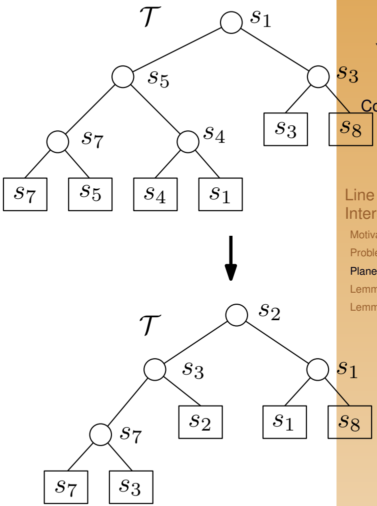
1. $U(p) \leftarrow$ segments whose upper endpoint is p;
2. Find all segments stored in $\mathcal{T}$ that contain p;
   $L(p) \leftarrow$ segments found whose lower endpoint is p;
   $C(p) \leftarrow$ segments found that contain p in their interior.
3. **if** $|L(p) \cup U(p) \cup C(p)| > 1$
4.   **then** Report p as an intersection, together with $L(p), U(p)$, and $C(p)$.
5. Delete the segments in $L(p) \cup C(p)$ from $\mathcal{T}$.
6. Insert the segments in $U(p) \cup C(p)$ into $\mathcal{T}$.
7. **if** $U(p) \cup C(p) == \emptyset$
8.   **then** s_l and $s_r \leftarrow$ the left and right neighbors of p in $\mathcal{T}$.
9.   FINDNEWEVENT(s_l, s_r, p)
10.   **else** $s' \leftarrow$ the leftmost segment of $U(p) \cup C(p)$ in $\mathcal{T}$.
    $s_l \leftarrow$ the left neighbor of s' in $\mathcal{T}$.
11.   FINDNEWEVENT(s_l, s', p)
12. $s'' \leftarrow$ the rightmost segment of $U(p) \cup C(p)$ in $\mathcal{T}$.
13. $s_r \leftarrow$ the right neighbor of s'' in $\mathcal{T}$.
14.   FINDNEWEVENT(s'', s_r, p)

Algorithm FindNewEvent(s_l, s_r, p)

1. **if** s_l and s_r intersect below the sweep line, or on it and to the right of the current event point p, and the intersection is not yet present as an event in $\mathcal{Q}$
2. **then** Insert the intersection point as an event into $\mathcal{Q}$.

Plane sweep algorithm





Plane sweep algorithm

Lemma 2.2

Algorithm FINDINTERSECTIONS computes all intersection points and the segments that contain it correctly.

Proof. (By induction on priority of points)

- p: An intersection point.
- **Induction hypothesis:** All intersection points $q \prec p$ have been computed correctly.
- **Claim:** p is computed correctly.
- Case 1: p is an endpoint of one or more of the segments.
  Easy!
- Case 2: p is NOT an endpoint.
  We show that p will be inserted into Q at some moment.
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Lemma 2.3

The running time of Algorithm FINDINTERSECTIONS is $\mathcal{O}((n+I)\log n)$, where $I = \#$ intersections.

Proof.

- Constructing the event queue on the segment endpoints: $\mathcal{O}(n \log n)$ time.
- Initializing the status structure: $\mathcal{O}(1)$ time.
- Deletions and insertions on $\mathcal{Q}$: $\mathcal{O}(\log n)$ time each.
- Insert, delete, and neighbor finding on $\mathcal{T}$: $\mathcal{O}(\log n)$ time each.
- $\#$oper. for p: $\mathcal{O}(m(p))$ $(m(p) :=$ card $(L(p) \cup U(p) \cup C(p)))$
- If $m = \sum_p m(p)$ then complexity= $\mathcal{O}(m \log n)$.
- $m = \mathcal{O}(n+k)$. $(k$: output size (points+lines)).

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The running time of Algorithm FINDINTERSECTIONS is $\mathcal{O}((n+I)\log n)$, where $I = \#$ intersections.

Proof.

- Constructing the event queue on the segment endpoints: $\mathcal{O}(n \log n)$ time.
- Initializing the status structure: $\mathcal{O}(1)$ time.
- Deletions and insertions on $\mathcal{Q}$: $\mathcal{O}(\log n)$ time each.
- Insert, delete, and neighbor finding on $\mathcal{T}$: $\mathcal{O}(\log n)$ time each.
- $\#$oper. for p: $\mathcal{O}(m(p))$ $(m(p) :=$ card $(L(p) \cup U(p) \cup C(p)))$
- If $m = \sum_p m(p)$ then complexity= $\mathcal{O}(m \log n)$.
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Plane sweep algorithm

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The running time of Algorithm FINDINTERSECTIONS is $\mathcal{O}((n+I)\log n)$, where $I = \#$ intersections.

Proof.

- (Claim:) $m = \mathcal{O}(n+I)$. (I : $\#$ intersections).
- Consider the segments as a planar graph.
- n_v : $\#$ vertices, n_e : $\#$ edges, n_f : $\#$faces
- $m(p) = degree(p) \Rightarrow m = \sum_p m(p) = 2n_e$.
- $n_v \leq 2n + I$, $n_f \leq 2n_e/3$
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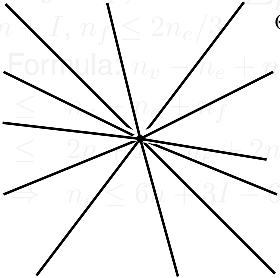

$\Theta(n)$ line segments

$k \in \Theta(n^2)$

$I \in \Theta(1)$

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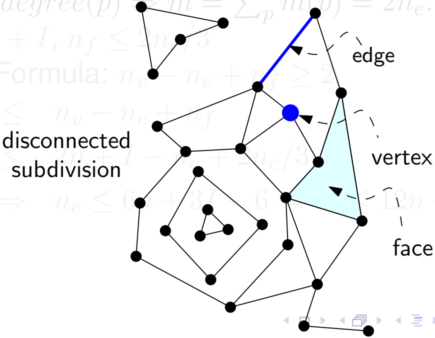

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- $\mathcal{O}(n + I)$ (size of the event queue)
- Can be improved to $\mathcal{O}(n)$: only store intersection points of pairs of segments that are currently adjacent on the sweep line.

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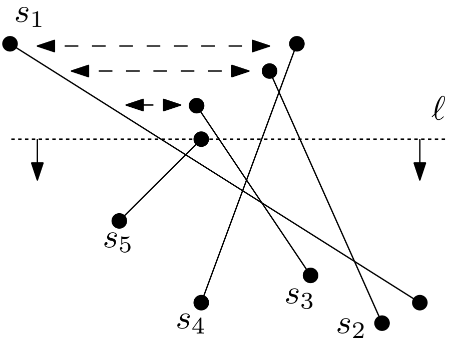

Yazd Univ.
Computational Geometry
Line Segment Intersection
Motivation
Problem
Plane sweep algorithm
Lemma 2.2
Lemma 2.3

Plane sweep algorithm

Theorem 2.4

Let S be a set of n line segments in the plane. All intersection points in S, with for each intersection point the segments involved in it, can be reported in $\mathcal{O}(n \log n + I \log n)$ time and $\mathcal{O}(n)$ space, where I is the number of intersection points.