A lower bound for computing geometric spanners

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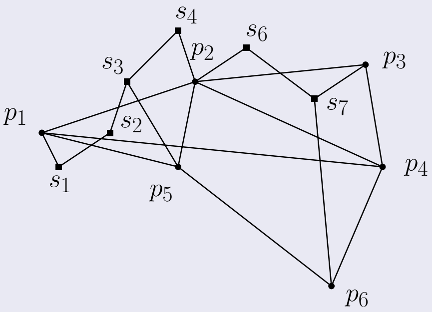
Outline

Geometric network

Definition

- Geometric network on $P \subset \mathbb{R}^d$
- The length of a path
- t-spanner path between two points
- t-spanner on P
- Steiner t-spanner on P



Spanners and a lower bound

Given $p = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$, where d is a constant, and a constant $t > 1$

A sparse t-spanner on P in $\mathcal{O}(n \log n)$,

Question: Compute a sparce t-spanner on P in $o(n \log n)$?

In 2001, Chen, Das, and Smid gave a lower bound $\Omega(n \log n)$ on a point set from $\mathbb{R}$ in the algebraic computation tree model

If $P \subset \mathbb{R}^d$, where $d > 1$

General position

Definition

General position for a point set $P \subset \mathbb{R}^d$

Conjecture: Let $d \geq 2$ be an integer constant. In the algebraic computation tree model, any algorithm that, given a set P of n points in $\mathbb{R}^d$ that are in general position, and a real number $t > 1$, constructs a Steiner t-spanner for P, takes $\Omega(n \log n)$ time in the worst case.

The algebraic computation tree model

Definition

computatiom problem: $A : \mathbb{R}^n \to S$, for example *sorting problem*

Definition

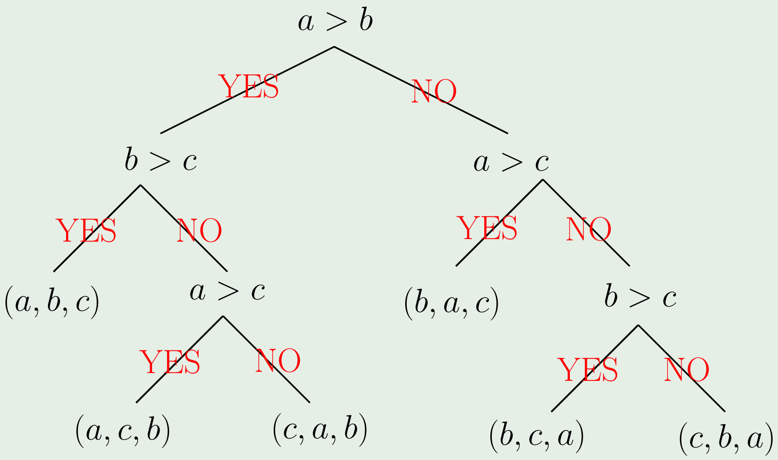
n: a positive integer, S: a solution space.
An algebraic computation-tree on $s_1, s_2, \ldots, s_n$ is a finite tree T:

- Each leaf is labeled with the combinatorial description, in terms of $s_1, s_2, \ldots, s_n$, of an element in S.
- Each node u having one child is labeled with $Z(u)$
  (a) $Z(u) := A_1 \& A_2$, where $\& \in \{+, -, \times, /\}$, or
  (b) $Z(u) := \sqrt{A_1}$,
  (i) $A_i = Z(u')$, (ii) $A_i \in \{s_1, \ldots, s_n\}$, or (iii) A_i is a constant.
- Each node u having two children of the form $A \leq 0$ or $A > 0$
  (i) $A_i = Z(u')$, (ii) $A_i \in \{s_1, \ldots, s_n\}$

The algebraic computation tree model

Example

Sort given (a, b, c)



The algebraic computation tree model

Definition

A computation problem $A : \mathbb{R}^n \to S$ is solvable in the algebraic computation tree model if exists a T that for any $(s_1, \ldots, s_n) \in \mathbb{R}^n$ returns $A(s_1, \ldots, s_n)$.

Definition

The time complexity of a problem

The unit gap decision problem

Definition

Given $S = (x_1, \ldots, x_n) \in \mathbb{R}^n$, if for all $i \neq j$, $|x_i - x_J| \geq 1$.

This computation problem has an $\Omega(n \log n)$ time complexity.

Theorem

Let W be any set in $\mathbb{R}^n$ and let $\mathcal{D}$ be any algorithm that accepts W. Let $\#W$ denotes the number of connected components of W. Then the worst-case running time of $\mathcal{D}$ in the algebraic computation tree model is $\Omega(\log \#W - n)$.

The time complexity of the unit gap problem

Lemma

The unit gap problem has an $\Omega(n \log n)$ time complexity.

Proof.

$(x_1, \ldots, x_i, \ldots, x_n) := (1, \ldots, i, \ldots, n)$
Let π and ρ be two distinct permutation of $1, \ldots, n$.
$p := (x_{\pi(1)}, \ldots, x_{\pi(n)})$, $r := (x_{\rho(1)}, \ldots, x_{\rho(n)})$,
There are $i \neq j$, $\pi(i) < \pi(j)$ and $\rho(i) > \rho(j)$,
Hence, any curve between p and r contains $q = (q_1, \ldots, q_n)$ s.t. $q_i = q_j$.
Hence, solution space of the problem has at least $n!$ differnt component.
Hence, the problem has an $\Omega(n \log n)$ time complexity. □

The reduction

Let STEINER-SPANNER be an algorithm,

Input: A set S of points in general position and $t > 1$.

Output: A Steiner t-spanner on S with $o(|S| \log |S|)$ and a label to each vertex as orginal or Steiner.

$S = \{x_1, \ldots, x_n\}$, $x_{min} = \min\{x_i\}$, $y_i := x_i + |x_{min}| + 1 \geq 1$,
$p_i := (y_i, y_i^2, \ldots, y_i^d)$.

Lemma

Points $P = \{p_i\}$ are in general position.

The reduction

$x_i \in S \rightarrow p_i \in \mathbb{R}^d$,

$\textsc{Steiner-Spanner}(P, 1.1)$

for each point p, traverse the graph in a breath-first search all points inside the ball C_{p_i} centered at p_i with radius $2|p_i p_i'|$, where $p_i' := ((y_i - 1), (y_i - 1)^2, \ldots, (y_i - 1)^d)$

If finds a $p_j \in C_{p_i}$ s.t $|(p_i)_1 (p_j)_1| < 1$ return NO, otherwise return YES

The reduction

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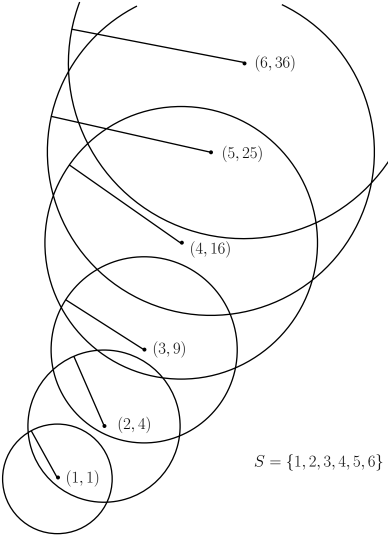

$S = \{1, 2, 3, 4, 5, 6\}$

Lemma

If the distance between the closest pair in the set of all input numbers of the reduction is greater than or equal to 1, then each ball C_p, $p \in P$, overlaps $\mathcal{O}(d)$ balls C_{p_i}.

Theorem

The time complexity of any algorithm for computing Steiner spanner in the algebraic computation tree model is $\Omega(n \log n)$.

For Further Reading

1. M. Ben-Or. Lower bounds for algebraic computation trees. In Proceedings of the Fifteenth Annual ACM Symposium on Theory of Computing, STOC '83, pages 80-86, New York, NY, USA, 1983. ACM.
2. D. Z. Chen, G. Das, and M. Smid. Lower bounds for computing geometric spanners and approximate shortest paths. Discrete Applied Mathematics, 110:151-167, 2001.
3. G. Narasimhan and M. Smid. Geometric spanner networks. Cambridge University Press, 2007.
4. F. P. Preparata and M. I. Shamos. Computational Geometry: An Introduction. Springer-Verlag, New York, USA, 1985.

Thanks!