Ortho-radial drawings of graphs

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Abstract

By an ortho-radial drawing of a graph we mean a planar drawing on concentric circles such that each edge is an alternating sequence of circular and radial segments, where a circular segment is a part of a circle and a radial segment is a part of a half-line starting at the center of the circles. Ortho-radial drawings are topologically an extension of orthogonal drawings to drawings on a cylinder.

We study the relationship between ortho-radial drawings and orthogonal drawings, then we prove necessary and sufficient conditions for a path, cycle or a theta graph to have an ortho-radial drawing consistent with a C-shape (cylindrical shape) which is a specification of the direction in which each edge must be drawn. Furthermore, we present an example of a C-shape of a graph such that all of its cycles have an ortho-radial drawing but the graph itself does not have any ortho-radial drawing with this C-shape. This is in contrast to the properties of orthogonal drawings on the plane.

1 Introduction

Orthogonal drawings are planar drawings of graphs in which every edge is represented by a chain of horizontal and vertical segments. Orthogonal drawings are of much interest because of their application in VLSI design and layout. An orthogonal shape, P-shape (planar-shape), of a graph specifies in which of the four horizontal or vertical directions each edge must be drawn [8]. Vijayan and Wigderson [8] found a necessary and sufficient condition for orthogonal shapes to have an orthogonal drawing. Many algorithms for orthogonal drawings are based on this condition [1, 5, 7].

Drawing of an orthogonal shape in three dimensional space is a challenging problem. Several studies have been conducted on this issue [2, 3, 4, 6].
We call a grid composed of concentric circles with center $S$ and half-lines starting at $S$ an ortho-radial grid. An ortho-radial drawing of a graph $G$ is a planar drawing of $G$ in an ortho-radial grid such that each edge is an alternating sequence of circular and radial segments where a circular segment is a connected part of a circle and a radial segment is a connected part of a half-line, not including $S$.

Since in an ortho-radial drawing no vertex is drawn at $S$ and no edge passes through $S$, an ortho-radial drawing is topologically a drawing in a grid on a cylinder or a drawing in a grid on a sphere in which no vertex is drawn on either the North pole or the South pole and no edge passes through either pole (see Figure 1).

In an ortho-radial grid, a directed radial segment has direction Down if it points towards $S$ and direction Up if it points away from $S$. A directed circular segment has direction Clockwise if $S$ is on its right side and direction Anticlockwise (Counterclockwise) if $S$ is on its left side (see Figure 2).

In a graph, the term dart is used for each of the two possible orientations $(u, v)$ and $(v, u)$ of an undirected edge $uv$.

Let $G$ be a graph with labels from the set $\{U (Up), D (Down), C (Clockwise), A (Anticlockwise)\}$ assigned to its darts. The assignment of labels is called a C-shape of $G$. Does an ortho-radial drawing of $G$ exist such that each dart is drawn as a single radial or circular segment with direction consistent with its associated label? We study this problem for some classes of graphs.

Note that the labels of the darts of $G$ imply a cyclic order of the darts leaving each vertex, and therefore imply a combinatorial embedding of $G$. We will only be concerned with the case that this embedding is planar. Vijayan and Wigderson [8] proved that a biconnected graph has an orthogonal drawing with a given P-shape.
if and only if its faces have orthogonal drawings with the induced P-shapes. This theorem is not true in the case of C-shapes. The graph shown in Figure 3(a) does not have any ortho-radial drawing with the C-shape shown but all its faces have ortho-radial drawings with the induced C-shapes. In the example in Figure 3(b), all the cycles of the graph have ortho-radial drawings, but the graph itself does not have an ortho-radial drawing with the given C-shape.

Figure 3: The examples which are in contrast to the properties of orthogonal drawings on the plane

The remainder of this paper is organized as follows: Section 2 presents the exact definition of C-shapes, the relationship between orthogonal and ortho-radial drawings, and necessary and sufficient conditions for paths and cycles to have ortho-radial drawings with given C-shapes. Section 3 deals with the problem for biconnected graphs and presents necessary and sufficient conditions for theta graphs. Section 4 summarizes the results and gives directions for further work.

2 Ortho-radial drawings and C-shapes

2.1 C-shapes

For an undirected graph $G$, a C-shape is a labeling of the darts of $G$ such that:

1. The labels are in the set $\{C, A, D, U\}$.
2. The labels of darts $(u, v)$ and $(v, u)$ are opposite. (The labels $C$, $A$, $D$ and $U$ are the opposites of $A$, $C$, $U$ and $D$, respectively.)
3. No two consecutive darts $(u, v)$ and $(v, w)$ have opposite labels.

Let $\gamma$ be a C-shape of $G$. A drawing of $G$ with C-shape $\gamma$ is an ortho-radial drawing $\Gamma$ of $G$ where each dart consists of a single radial or circular segment with direction consistent with its label. A C-shape $\gamma$ of a graph $G$ is called drawable if there is a drawing of $G$ with C-shape $\gamma$.

Let $C = v_1, v_2, ..., v_{n+1} = v_1$ be a cycle with C-shape $\gamma$ and $\sigma = \sigma_1\sigma_2...\sigma_n$ be the labels of darts $(v_1, v_2), (v_2, v_3), ..., (v_n, v_{n+1})$. We call $\sigma$ a C-shape cycle. For a path $p = v_1, v_2, ..., v_{n+1}$ with C-shape $\gamma$, let $\sigma = \sigma_1\sigma_2...\sigma_n$ be the labels of darts $(v_1, v_2), (v_2, v_3), ..., (v_n, v_{n+1})$. We call $\sigma$ a C-shape path. If $\sigma$ is a C-shape path or
C-shape cycle, a dart \((v_i, v_{i+1})\) for \(1 \leq i \leq n\) is a forward dart of \(\sigma\). We say a path \(p\) is labeled with \(X\) if all of its forward darts have the same label \(X\).

For a C-shape path or cycle \(\sigma = \sigma_1\sigma_2...\sigma_n\), we define \(\overline{\sigma} = \overline{\sigma_n}\overline{\sigma_{n-1}}...\overline{\sigma_1}\) where \(\overline{\sigma_i}\) is the opposite of \(\sigma_i\).

If \(e_1 = (u, v)\) and \(e_2 = (v, w)\), \(u \neq w\), are two darts of a graph \(G\) with a given C-shape, the angle between these darts is the angle \((\pi/2, \pi\) or \(3\pi/2)\) on the left when we move from \(u\) to \(w\). Let \(l_1\) and \(l_2\) be the labels of \(e_1\) and \(e_2\). The function \(\text{turn}(l_1, l_2)\) is defined as follows:

\[
\text{turn}(l_1, l_2) = \begin{cases} 
1 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \frac{\pi}{2} \\
0 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \pi \\
-1 & \text{if the angle between } e_1 \text{ and } e_2 \text{ is } \frac{3\pi}{2}
\end{cases}
\]  

(1)

Let \(\sigma = \sigma_1\sigma_2...\sigma_n\) be a C-shape path. The rotation of \(\sigma\), \(\text{rot}(\sigma)\), is defined as follows:

\[
\text{rot}(\sigma) = n^{-1} \sum_{i=1}^{n-1} \text{turn}(\sigma_i, \sigma_{i+1})
\]  

(2)

and for a C-shape cycle \(\sigma = \sigma_1\sigma_2...\sigma_n\), the rotation of \(\sigma\) is

\[
\text{rot}(\sigma) = \sum_{i=1}^{n-1} \text{turn}(\sigma_i, \sigma_{i+1}) + \text{turn}(\sigma_n, \sigma_1).
\]  

(3)

For a graph \(G\) with C-shape \(\gamma\), without loss of generality we can contract two consecutive darts with the same label into a single dart if the degree of their common vertex is two.

### 2.2 Ortho-radial and orthogonal drawings

In an ortho-radial drawing of a graph two cases may happen:

1. The point \(S\) is in the external face. This is a type-1 ortho-radial drawing (see Figure 4(a)).

2. The point \(S\) is in an internal face. This is a type-2 ortho-radial drawing (see Figure 4(b)).

![Figure 4: (a) A type-1 ortho-radial drawing, (b) A type-2 ortho-radial drawing](image)

There is a close relationship between orthogonal drawings and type-1 ortho-radial drawings. We present this relationship formally in the next theorem.
**Theorem 1** Every type-1 ortho-radial drawing of a graph $G$ can be transformed into an orthogonal drawing in the plane in such a way that each vertical segment becomes a radial segment and each horizontal segment becomes a circular segment, and vice versa.

![Orthogonal drawing and corresponding type-1 ortho-radial drawing](image)

Figure 5: An orthogonal drawing and a corresponding type-1 ortho-radial drawing

In an ortho-radial drawing, a *bend* is a point at which an edge changes direction. Theorem 1 proves that for each orthogonal drawing there is an ortho-radial drawing with the same number of bends (see Figure 5). In fact, there are some graph families whose orthogonal drawings have at least a linear number of bends, but have ortho-radial drawings with no bends. Figure 6 illustrates an example of such a graph family. Since minimizing the number of bends is an important aesthetic criterion, ortho-radial drawings can be of more importance than just a new style of graph drawing.

Note that every graph with an orthogonal drawing or ortho-radial drawing has maximum degree at most 4.

![Example graphs](image)

Figure 6: This family of graphs has ortho-radial drawings with no bends, but no orthogonal drawings with less than a linear number of bends.

The formal definition of P-shapes is the same as that of C-shapes, the difference being that in P-shapes the labels are in the set \{L, R, D, U\}, where L, R, D and U are abbreviations for Left, Right, Down and Up, respectively (see [8]).

To state the main results of this paper, we need to present some results about P-shapes. We regard the *contour* of an internal face of a biconnected plane graph as an anticlockwise cycle formed by the edges on the boundary of the face, and we regard the *contour* of the external face of a biconnected plane graph as a clockwise cycle formed by the edges on the boundary of the face. The rotation of a P-shape cycle (path) is defined similarly to that of C-shapes [1]. This is a useful property of the rotation:
Property 1. [1]. If $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ is the P-shape of a face $f$ in an orthogonal drawing of a biconnected graph, then
\[
\text{rot}(\sigma) = \begin{cases} 
4 & \text{if } f \text{ is an internal face} \\
-4 & \text{if } f \text{ is the external face}
\end{cases}
\] (4)

In the following we provide two lemmas which are powerful tools for proving the main results of this paper. For a plane graph $G$, we denote by $C_0$ the contour of the external face.

**Lemma 1** Let $G$ be a biconnected plane graph with a drawable P-shape $\gamma$. Suppose $C_0$ contains three consecutive paths $p_1, p_2$ and $p_3$ labeled with $U$, $R$ and $D$ ($D$, $L$ and $U$), respectively. Then there is a drawing of $G$ where the vertices of $p_2$ have the largest (smallest) $y$-coordinate.

**Proof:** Let $u$ and $v$ be the first and last vertex of path $p_2$, respectively. We construct a graph $G'$ from $G$ by inserting a path $q$ with length 5 from $v$ to $u$ in the external face of $G$. Define $\gamma'$ to be the P-shape of $G'$ with the darts of $q$ labeled with $R$, $D$, $L$, $U$ and $R$, respectively and other darts labeled with the same label as in $\gamma$. It is easy to check that $\gamma'$ is a drawable P-shape and in every drawing of $G'$ with P-shape $\gamma'$, the vertices of $p_2$ have the largest $y$-coordinate. So, by removing the path $q$ from a drawing of $G'$, we reach a drawing of $G$ where the vertices of $p_2$ have the largest $y$-coordinate. \(\square\)

Let $G$ be a plane graph with a drawable P-shape $\gamma$, let $p_1$ be a maximal path labeled by $R$ on $C_0$ and let $p_2$ be a maximal path labeled by $L$ on $C_0$. We say $p_1$ and $p_2$ are admissible if $G$ has a drawing in which vertices of $p_1$ have the largest $y$-coordinate and vertices of $p_2$ have the smallest $y$-coordinate. In Figure 7, $p_1$ and $p_3$ are admissible, but $p_1'$ and $p_3$ are not admissible.

![Figure 7](image-url)

Figure 7: Paths $p_1$ and $p_3$ are admissible, but $p_1'$ and $p_3$ are not admissible.

**Lemma 2** Let $G$ be a plane graph with a drawable P-shape $\gamma$. Suppose that $p_1$, $p_2$, $p_3$ and $p_4$ are four paths of $C_0$ such that $C_0 = p_1 p_2 p_3 p_4$, $p_1$ is a maximal path labeled with $R$ and $p_3$ is a maximal path labeled with $L$, the first and last darts of $p_2$ are labeled with $D$ and the first and last darts of $p_4$ are labeled with $U$. Let $\sigma_2$ and $\sigma_4$ be the P-shapes of $p_2$ and $p_4$, respectively. Then $p_1$ and $p_3$ are admissible if and only if $\text{rot}(\sigma_2) = \text{rot}(\sigma_4) = 0$. 
Proof: Suppose that \( p_1 \) and \( p_3 \) are admissible. Let \( \Gamma \) be a drawing of \( G \) in which vertices of \( p_1 \) have the largest y-coordinates and vertices of \( p_3 \) have the smallest y-coordinates. By inserting two paths with lengths three in the external face as in Figure 8, we can conclude \( \text{rot}(\sigma_2) = \text{rot}(\sigma_4) = 0 \).

![Figure 8: Illustration of the proof of Lemma 2](image)

Suppose that \( \text{rot}(\sigma_2) = \text{rot}(\sigma_4) = 0 \). Let \( u \) and \( v \) be the first and the last vertices of \( p_1 \) and let \( z \) and \( w \) be the first and the last vertices of \( p_3 \). \( G' \) is the graph obtained from \( G \) by inserting two directed paths \( q_1 \) and \( q_2 \), \( q_1 \) with length 3 from \( v \) to \( z \) and \( q_2 \) with length 3 from \( w \) to \( u \). Let \( \gamma' \) be the P-shape obtained from \( \gamma \) by labeling darts of \( q_1 \) with labels \( R, D \) and \( L \) and darts of \( q_2 \) with labels \( L, U \) and \( R \), respectively. It is easy to check that \( \gamma' \) is a drawable P-shape and in every drawing of \( G' \) with P-shape \( \gamma' \), vertices of \( p_1 \) have the largest y-coordinate and vertices of \( p_3 \) have the smallest y-coordinate. By removing the paths \( q_1 \) and \( q_2 \) from a drawing of \( G' \), we obtain a drawing of \( G \) where vertices of \( p_1 \) have the largest y-coordinate and vertices of \( p_3 \) have the smallest y-coordinate.

\[ \square \]

2.3 Drawing of C-shape paths and C-shape cycles

In this section we prove necessary and sufficient conditions for a C-shape path or cycle to have an ortho-radial drawing.

By Theorem 1 and the fact that every P-shape path is drawable, we can conclude that every C-shape path is drawable. Theorem 1 and Property 1 prove that a C-shape cycle \( \sigma \) has a type-1 drawing if and only if \( \text{rot}(\sigma) = \pm 4 \). Theorem 2 characterizes C-shape cycles with type-2 drawings.

**Theorem 2** A C-shape cycle \( \sigma \) has a type-2 drawing if and only if \( \text{rot}(\sigma) = 0 \) and one of the following cases happens:

1. All the labels of \( \sigma \) are \( C \) or all the labels are \( A \).

2. \( \sigma \) contains at least one \( D \) and one \( U \) label.

**Proof:** Suppose that \( \sigma \) has a type-2 drawing \( \Gamma \) and \( \sigma \) is the C-shape of the external face of \( \Gamma \). (If \( \sigma \) is the C-shape of the internal face then we apply the following argument for \( \bar{\sigma} \).)

If all the labels of \( \sigma \) are the same, then all of them are \( A \) or \( C \) and \( \text{rot}(\sigma) = 0 \).
Otherwise, if \((u, v)\) is a dart labeled with \(D\) then in any drawing of \(\sigma\), \(v\) is closer to \(S\) than \(u\) is. Clearly the cycle can not be completed unless a dart labeled \(U\) is also present.

Now we will show that \(\text{rot}(\sigma) = 0\). Let \(e = (u, v)\) be the forward dart of \(\sigma\) corresponding to the farthest circular segment of \(\Gamma\) from the point \(S\). Without losing generality we can suppose that there is no other edge in the circle where \(e\) is drawn.

Since \(\sigma\) is the C-shape of the external face of \(\Gamma\), the darts before and after \(e\) have labels \(U\) and \(D\) respectively, and \(e\) has label \(C\). Thus, \(\sigma = \sigma'UAD\). If we remove \(e\) and add an anticlockwise circular segment from \(u\) to \(v\), then the resulting drawing is a type-1 drawing \(\Gamma'\) of \(\tau = \sigma'UAD\) in which \(\tau\) is the C-shape of the internal face of \(\Gamma'\) (see Figure 9). By Theorem 1 and Property 1, \(\text{rot}(\tau) = 4\). Therefore, \(\text{rot}(\sigma)\) and \(\text{rot}(\tau)\) satisfy the following equality:

\[
\text{rot}(\tau) = \text{rot}(\sigma) - \text{turn}(U, C) - \text{turn}(C, D) + \text{turn}(U, A) + \text{turn}(A, D)
\]

\[
= \text{rot}(\sigma) - (-1) - (-1) + 1 + 1.
\]

Hence \(\text{rot}(\sigma) = 0\).

Conversely, suppose that \(\sigma\) is a C-shape cycle such that \(\text{rot}(\sigma) = 0\). If all labels of \(\sigma\) are the same then \(\sigma\) has a type-2 drawing which is a circle. Otherwise, since \(\sigma\) has at least one \(D\) and one \(U\) by the third property in the definition of C-shapes, it contains some labels \(C\) or \(A\). Thus, \(\sigma\) contains some C-shape paths \(UAD\) or C-shape paths \(UCD\). If \(\sigma\) contains a C-shape path \(UAD\), then we can consider \(\sigma = \sigma'UAD\). Let \(\tau = \sigma'UCD\) then,

\[
\text{rot}(\tau) = \text{rot}(\sigma) - \text{turn}(U, A) - \text{turn}(A, D) + \text{turn}(U, C) + \text{turn}(C, D)
\]

\[
= 0 - 1 - 1 + (-1) + (-1) = -4.
\]

So \(\tau\) has a type-1 drawing with \(\tau\) as the C-shape of the external face. Let \((u, v)\) be the dart labeled with \(C\). By Lemma 1 and Theorem 1, \(\tau\) has a drawing \(\Gamma\) such that \(e\) is the farthest segment of \(\Gamma\) from the point \(S\). If we remove \(e\) and add an anticlockwise radial segment from \(u\) to \(v\), then the resulting drawing is a type-2 drawing of \(\sigma\) where \(\sigma\) is the C-shape of the internal face.

In the last case, if \(\sigma\) contains no C-shape path \(UAD\), it contains a C-shape path \(UCD\). \(\bar{\sigma}\) is a C-shape cycle containing C-shape path \(UAD\) and \(\text{rot}(\bar{\sigma}) = 0\). By the above discussion \(\bar{\sigma}\) is a drawable C-shape and there is a drawing \(\Gamma\) of \(\bar{\sigma}\) with \(\bar{\sigma}\) as the C-shape of the internal face. In \(\Gamma\), \(\sigma\) is the C-shape of the external face of \(\Gamma\). Hence \(\sigma\) has a type-2 drawing. This completes the proof. \(\square\)
Let $\sigma$ be a drawable C-shape cycle such that $\text{rot}(\sigma) = 0$. Then $\sigma$ has a drawing such that $\sigma$ is the C-shape of the external face if and only if $\sigma$ contains a C-shape path $UCD$ or all its labels are $C$, and $\sigma$ has a drawing such that $\sigma$ is the C-shape of the internal face if and only if $\sigma$ contains a C-shape path $UAD$ or all its labels are $A$. If there is a drawing $\Gamma$ where $\sigma$ is the C-shape of the external face of $\Gamma$, we call $\sigma$ an $E$-shape cycle (external-shape cycle), and if there is a drawing $\Gamma$ where $\sigma$ is the C-shape of the internal face of $\Gamma$, we call $\sigma$ an $I$-shape cycle (inner-shape cycle). Note that a drawable C-shape cycle with rotation 0 is either an $I$-shape cycle or an $E$-shape cycle and it is possible that it is both.

3 Ortho-radial drawings of biconnected graphs with given C-shapes

A connected graph $G$ is biconnected if the removal of each vertex does not make it disconnected. A theta graph is a biconnected graph consisting of three disjoint paths of length at least two between two vertices of degree three.

Let $G$ be a graph with C-shape $\gamma$. The labels of $\gamma$ define a cyclic order of the darts leaving each vertex; that is, they define a combinatorial embedding of the graph on some surface. We call this the embedding induced by $\gamma$. This is a planar embedding if the number of faces obeys Euler's formula.

The next theorem is a clarification of Theorem 5.1 of [8], where the need for planarity of the induced embedding is not clearly stated.

**Theorem 3** Let $G$ be a biconnected graph with at least three edges. A P-shape $\gamma$ of $G$ is drawable if and only if the embedding induced by $\gamma$ is a planar embedding and the P-shape of each face is drawable.

Let $G$ be a biconnected graph with at least three edges and let $\gamma$ be a C-shape of $G$. The proof of necessity in Theorem 3 is true for $\gamma$. In fact, if $\gamma$ is drawable then the embedding induced by $\gamma$ is a planar embedding and the C-shape of each face is drawable. Also, if the C-shapes of all faces of $G$ are drawable and the rotation in every face is 4 other than one whose rotation is -4, then by Theorem 3 and Theorem 1, $\gamma$ is drawable and $G$ has a type-1 ortho-radial drawing. The following theorem formally states the necessary condition for C-shape $\gamma$ to be drawable.

**Theorem 4** If $\gamma$ is a drawable C-shape, $F_1, F_2, ..., F_k$ are the faces of $G$ and $\sigma_i, 1 \leq i \leq k$, is the C-shape cycle of $F_i$ then one of the following happens:

1. For some face $F_j, \text{rot}(\sigma_j) = -4$ and $\text{rot}(\sigma_i) = 4$ for the other faces. In this case $\gamma$ has a type-1 drawing.

2. $\text{rot}(\sigma_j) = \text{rot}(\sigma_i) = 0$, for some $1 \leq j \neq l \leq k$, $\text{rot}(\sigma_l) = 4$ for other faces, one of $\sigma_j$ and $\sigma_l$ is an $I$-shape cycle and the other is an $E$-shape cycle. In this case $\gamma$ has a type-2 drawing.
The example in Figure 3(a) shows that Theorem 3 does not hold in the case of C-shapes. It is easy to check that the C-shapes of all its faces are drawable and it has a cycle whose C-shape is not drawable. In the example in Figure 3(b), even the C-shapes of all the cycles of the graph are drawable, but by Theorem 5 we can prove that the C-shape of the graph itself is not drawable.

For a path \( p = v_1, \ldots, v_k \) denote by \( \bar{p} \) the reverse path \( v_k, \ldots, v_1 \). By removing \( p \), we mean removing all the edges of \( p \) and all the vertices of \( p \) other than end-vertices.

Let \( T \) be a theta graph with three paths \( p_1, p_2 \) and \( p_3 \) connecting vertex \( p \) to vertex \( q \). Let \( \sigma \) be a C-shape of \( T \) which induces a planar embedding of \( T \), and let \( \sigma_i \) be the C-shape path of \( p_i \) induced by \( \sigma \), \( i = 1, 2, 3 \). We denote by \( \tau_1 \), \( \tau_2 \) and \( \tau_3 \) the C-shape cycles \( \sigma_1 \bar{p}_3, \sigma_2 \bar{p}_1 \) and \( \sigma_3 \bar{p}_2 \), respectively. The next theorem presents necessary and sufficient conditions for \( \sigma \) to have a type-2 ortho-radial drawing.

**Theorem 5** Considering the above definitions, \( T \) has a type-2 ortho-radial drawing with C-shape \( \sigma \) if and only if the C-shape cycles \( \tau_1, \tau_2 \) and \( \tau_3 \) are drawable, the rotation of one of them, say \( \tau_2 \), is 4, one of them, say \( \tau_3 \), is an I-shape cycle and the other one, \( \tau_1 \), is an E-shape cycle, and one of the following cases happens.

1. All the labels of \( \tau_3 \) are A.
2. All the labels of \( \tau_1 \) are C.
3. \( \tau_1 \) has a C-shape path \( UCD \) such that at least one of the darts of the subpath labeled by \( C \) is on \( p_3 \).
4. \( \tau_3 \) has a C-shape path \( DAU \) such that at least one of the darts of the subpath labeled by \( A \) is on \( p_3 \).
5. \( \sigma_2 \) has a C-shape path \( DCU \) and \( \bar{\sigma}_1 \) has a C-shape path \( UAD \) such that \( \tau_2 \) is \( \tau_21DCU\tau_22UAD\tau_23 \) for some C-shape paths \( \tau_21, \tau_22 \) and \( \tau_23 \), and \( \text{rot}(D\tau_23\tau_21D) = \text{rot}(U\tau_22U) = 0 \).

**Proof:** Suppose that \( T \) has a type-2 ortho-radial drawing \( \Gamma \). Let \( \tau_1, \tau_2 \) and \( \tau_3 \) be the C-shape of the external face, the internal face not containing point \( S \) and the internal face containing point \( S \), respectively. Thus, \( \tau_1 \) is an E-shape cycle, \( \text{rot}(\tau_2) = 4 \) and \( \tau_3 \) is an I-shape cycle. Suppose none of the cases 1, 2, 3 and 4 happens. Let \( r \) and \( r' \) be the nearest and farthest segments of \( \Gamma \) from \( S \). Then \( r \) is a subpath of cycle \( p_3\bar{p}_2 \) labeled by \( A \) and the edges before and after \( r \) are labeled by \( D \) and \( U \) respectively, and \( r' \) is a subpath of cycle \( p_1\bar{p}_3 \) labeled by \( C \) with the edges before and after \( r' \) are labeled by \( U \) and \( D \) respectively. Since cases 3 and 4 do not happen, \( r \) and \( r' \) are respectively on \( p_2 \) and \( p_1 \) which are therefore the nearest and farthest segments from \( S \) on cycle \( p_2\bar{p}_1 \) in \( \Gamma \). Thus, there are some C-shape paths \( \tau_21, \tau_22 \) and \( \tau_23 \) such that \( \tau_2 = \tau_21DCU\tau_22UAD\tau_23 \) and by considering the corresponding orthogonal drawing of \( p_2\bar{p}_1 \), Lemma 2 and Theorem 1, we conclude \( \text{rot}(D\tau_23\tau_21D) = \text{rot}(U\tau_22U) = 0 \). This proves the necessity.

Now we prove the sufficiency for each case. In the first case, it is easy to see that the first and last dart of \( \bar{p}_1 \) are respectively labeled by \( U \) and \( D \). Thus, by considering
the corresponding orthogonal drawing of cycle $p_2\overline{p_1}$, Lemma 1 and Theorem 1, there is a drawing of cycle $p_2\overline{p_1}$ with C-shape cycle $\tau_2$ such that vertices $p$ and $q$ are drawn on the smallest cycle of the ortho-radial grid. Without loss of generality, we can suppose that there are no other vertices on that cycle except the vertices of path $p_2$. By inserting an anticlockwise circular segment from $p$ to $q$, we obtain a drawing of $T$. For the second case, we can use a similar approach to obtain a drawing of $T$.

Figure 10: The number written in each face is the rotation of the C-shape cycle corresponding to the face

Suppose $\tau_1$ has a C-shape path $UCD$ such that at least one of the darts of the subpath $r = v_1, ..., v_k$, labeled by $C$ is on $\overline{p_3}$. If for some $i_p$ and $i_q$, $1 < i_p < i_q < k$, $p = v_{i_p}$ and $q = v_{i_q}$ (see Figure 10(a)), then the first and last darts of $p_2$ are labeled by $D$ and $U$ respectively. Vertices $p$ and $q$ divide $r$ into three subpaths, $q_1 = v_1, ..., v_{i_p}$, $q_2 = v_{i_p}, ..., v_{i_q}$ and $q_3 = v_{i_q}, ..., v_k$. Remove $q_1$ and $q_2$, add edge $v_kv_1$ and label dart $(v_k, v_1)$ by $C$. The resulting graph consists of two disjoint cycles which have type-1 ortho-radial drawings. By considering corresponding orthogonal drawings of the cycles, Lemma 1 and Theorem 1 show that there are coincidence drawings of the cycles such that $v_1v_k$ and $q_2$ are drawn as the farthest segments from $S$. The drawing can be easily converted to a drawing of $T$ with C-shape $\sigma$. Otherwise, let $v_j$ be the first vertex of $r$ which is on $\overline{p_3}$ and let $v_{j'}$ be the last vertex of $r$ on $\overline{p_3}$ such that all the darts of $r$ which are between $v_j$ and $v_{j'}$ are on $\overline{p_3}$. Remove the subpath of $r$ which is between $v_j$ and $v_{j'}$, then add edge $v_kv_1$ and label dart $(v_k, v_1)$ by $C$ (see Figure 10(b)). The resulting graph $T'$ is a theta graph. Let $\sigma'$ be the C-shape of $T'$. Computing the rotation of C-shape cycles corresponding to the faces proves that $T'$ has a type-1 ortho-radial drawing. Let $r''$ be the path in $T'$ consisting of $v_kv_1$ and the remaining part of $r$. The P-shape corresponding to $\sigma'$ satisfies the conditions of Lemma 1 and so by Theorem 1, $T'$ has a type-1 drawing such that $r''$ is drawn as the farthest segment from $S$ in the drawing. This is easily converted into
a type-2 ortho-radial drawing of graph $T$ with C-shape $\sigma$. A similar approach works for case 4.

In order to prove the sufficiency in case 5, let $r$ and $r'$ be the subpaths labeled with $C$ and $A$, respectively. By replacing the labels of the darts of $r$ and $r'$ with $A$ and $C$ respectively, we obtain a drawable C-shape $\sigma'$ of $T$ which has a type-1 ortho-radial drawing (see Figure 10(c)). By considering the corresponding orthogonal drawing of $T$, Lemma 2 and Theorem 1 shows that $T$ has an ortho-radial drawing with C-shape $\sigma'$ such that $r$ and $r'$ are the nearest and farthest paths from $S$ in the drawing and this drawing can be easily transformed to a type-2 ortho-radial drawing of $T$ with C-shape $\sigma$. □

4 Concluding remarks

The problem of determining C-shapes that have an ortho-radial drawing is still open. The solution of this problem is a suitable base for studying ortho-radial drawings. Since there are some graphs that have ortho-radial drawings with fewer bends than orthogonal drawings, the problem of computing ortho-radial drawings of graphs with a minimum number of bends is an interesting problem.

References


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