ON THE CONTINUITY OF PRINCIPAL EIGENVALUES FOR BOUNDARY VALUE PROBLEMS WITH INDEFINITE WEIGHT FUNCTION WITH RESPECT TO RADIUS OF BALLS IN R^N.

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ABSTRACT. We investigate the continuity of principal eigenvalues (i.e., eigenvalues corresponding to positive eigenfunctions) for the boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda g(x)u(x) & x \in B_R(0) \\ u(x) = 0 & |x| = R \end{cases}$$

,where $B_R(0)$ is a ball in R^N, and g is a smooth function, and we show that $\lambda_1^+(R)$ and $\lambda_1^-(R)$ are continuous functions of R.

1. INTRODUCTION

We study the function $R \to \lambda_1^+(R)$ where $\lambda_1^+(R)$ being the unique positive principal eigenvalue (i.e., eigenvalue corresponding to positive eigenfunction) for the boundary value problem

$$\begin{cases} -\Delta u(x) = \lambda g(x)u(x) & x \in B_R(0) \\ u(x) = 0 & |x| = R \end{cases} \tag{1}$$

where Δ is the standard Laplace operator, $B_R(0)$ is a ball in R^N, and $g : B_R(0) \to R$ is a smooth function with changes sign on $B_R(0)$.
In recent years there has been interest in such problems since Fleming [1] studied the following equation which arises in population genetics

$$u_t(x,t) = \Delta u + \lambda g(x) f(u), \qquad x \in D \tag{2}$$

where D is a bounded domain with smooth boundary, g changes sign on D and f is some function of class C^1 such that $f(0) = 0 = f(1)$.
Fleming,s results suggested that nontrivial steady-state solutions were bifurcating off the trivial solutions $u \equiv 0$ and $u \equiv 1$. In order to investigate these bifurcation phenomena it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\Delta u(x) = \lambda g(x) u(x) \qquad x \in D \tag{3}$$

The ordinary differential equation versions of (3) were studied by Sturm, Picone[2] and Bocher[3]. Motivated by Fleming,s paper Brown and Lin[4] and Hess and Kato[5] studied the eigenvalues and eigenfunctions of (3) in the partial differential equation case. Since in population genetics the unknown function u represents the frequency of a population only solutions $u \geq 0$ are of interest.
In order that nontrivial solutions bifurcating off the zero solution are positive it is necessary that the eigenfunction of the corresponding eigenvalue is positive. Such eigenvalues and eigenfunctions are called principal eigenvalues and eigenfunctions.
The existence of principal eigenvalues of (1) has been studied previously in [4,5,6]. It is well known (see[5]) that there exists a double sequence of eigenvalues for (1)

$$\ldots \lambda_2^- < \lambda_1^+ < 0 < \lambda_1^+ < \lambda_2^+ < \ldots$$

$\lambda_1^+(\lambda_1^-)$ being the unique positive(negative) principal eigenvalue, i.e., (1) has solution $u(v)$ which are positive in $B_R(0)$, and we call principal eigenfunction corresponding to principal eigenvalue $\lambda_1^+(\lambda_1^-)$.

The variational characterizations of $\lambda_1^+(R)$ and $\lambda_1^-(R)$ are proved in [8],

Theorem 1. We have

$$\lambda_1^+(R)=\inf\{\frac{\int_{B_R(0)}|\nabla u|^2dx}{\int_{B_R(0)}g(x)u^2dx}:u\in H_0^1(B_R(0)),\int_{B_R(0)}g(x)u^2dx>0\}$$

and

$$\lambda_1^-(R)=\sup\{\frac{\int_{B_R(0)}|\nabla u|^2dx}{\int_{B_R(0)}g(x)u^2dx}:u\in H_0^1(B_R(0)),\int_{B_R(0)}g(x)u^2dx<0\}.$$

Also it is proved that (see [8])

Theorem 2. $\lambda_1^+(R)$ can be characterized as

$$\lambda_1^+(R)=\inf\{\frac{\int_{B_R(0)}|\nabla u|^2dx}{\int_{B_R(0)}g(x)u^2dx}:u\in C_0^\infty(B_R(0)),\int_{B_R(0)}g(x)u^2dx>0\},$$

and similarly for $\lambda_1^-(R)$.

2. On the continuity of $\lambda_1^+(R)$ and $\lambda_1^-(R)$ with respect to R

First we prove that $\lambda_1^+(R)(\lambda_1^-(R))$ is a strictly decreasing(increasing) function of R

Theorem 3. $R\to\lambda_1^+(R)$ is a strictly decreasing function.

Proof. It is proved in [8] that $R\to\lambda_1^+(R)$ is a decreasing function of R, so it is sufficient to show the strictly of it. We prove it by a contradiction argument. On the contrary, suppose there exists R and R'

such that $R < R'$ but $\lambda_1^+(R) = \lambda_1^+(R')$. Then there exist two positive functions u and v on $B_R(0)$ and $B_{R'}(0)$, respectively, such that

$$\begin{cases} -\Delta u(x) = \lambda_1^+(R)g(x)u(x) & x \in B_R(0) \\ u(x) = 0 & |x| = R \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta v(x) = \lambda_1^+(R')g(x)v(x) & x \in B_{R'}(0) \\ v(x) = 0 & |x| = R' \end{cases} \tag{5}$$

From (5) we have

$$\begin{cases} -\Delta v(x) = \lambda_1^+(R')g(x)v(x) & x \in B_R(0) \\ v(x) > 0 & |x| = R \end{cases} \tag{6}$$

Multiplying (4) by v and integrating over $B_R(0)$ we obtain

$$\int_{B_R(0)} \nabla u(x)\nabla v(x)dx - \int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds = \lambda_1^+(R)\int_{B_R(0)} g(x)u(x)v(x)dx \tag{7}$$

also by multiplying (6) by u and integrating over $B_R(0)$ we obtain

$$\int_{B_R(0)} \nabla u(x)\nabla v(x)dx - \int_{|x|=R} u(x)\frac{\partial v}{\partial n}(x)ds = \lambda_1^+(R')\int_{B_R(0)} g(x)u(x)v(x)dx \tag{8}$$

Now by subtracting (8) from (7) we obtain

$$-\int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds = [\lambda_1^+(R) - \lambda_1^+(R')]\int_{B_R(0)} g(x)u(x)v(x)dx \tag{9}$$

By (4) and (6) we have

$$\int_{|x|=R} v(x)\frac{\partial u}{\partial n}(x)ds < 0$$

and so by (9) we obtain $\lambda_1^+(R) - \lambda_1^+(R') \neq 0$, and this is a contradiction.
$\square$

Also by a similar argument we can obtain

Theorem 4. $\lambda_1^-(R)$ is a strictly increasing function of R.

Theorem 5. $R \to \lambda_1^+(R)$ is a continuous function of R.

Proof. Let $\epsilon > 0$ is given. Let $R_1 > R$ and sufficiently close to R, it is enough to show that

$$\lambda_1^+(R) < \lambda_1^+(R_1) + \epsilon.$$

Let $\varphi_1 \in H_0^1(B_{R_1}(0))$ is such that

$$\begin{cases} -\Delta\varphi_1(x) = \lambda_1^+(R_1)g(x)\varphi_1(x) & x \in B_{R_1}(0) \\ \varphi_1(x) = 0 & |x| = R_1 \end{cases}$$

We define $y = \frac{R_1}{R}x$ and $\hat{\varphi}(x) = \varphi_1(y)$ for $x \in B_R(0)$. We have $\hat{\varphi} \in H_0^1(B_R(0))$ and we have

$$\begin{aligned}
& \left|\int_{B_{R_1}(0)} g(x)\varphi_1^2(x)dx - \int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx\right| \\
= & \left|\int_{B_R(0)} g(x)\varphi_1^2(x)dx - \int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx + \int_{B_{R_1}(0)-B_R(0)} g(x)\varphi_1^2(x)dx\right| \\
\le & \left|\int_{B_R(0)} g(x)\varphi_1^2(x)dx - \int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx| + |\int_{B_{R_1}(0)-B_R(0)} g(x)\varphi_1^2(x)dx\right| \\
\le & \int_{B_R(0)} |g(x)||\varphi_1^2(x) - \hat{\varphi}^2(x)|dx + \int_{B_{R_1}(0)-B_R(0)} |g(x)\varphi_1^2(x)|dx \\
\le & \left(\int_{B_R(0)} |\varphi_1^2(x) - \varphi_1^2(\frac{R_1}{R}x)|dx\right) \sup_{x\in B_R(0)} |g(x)| \\
+ & |B_{R_1}(0) - B_R(0)| \sup_{x\in B_{R_1}(0)-B_R(0)} |g(x)\varphi_1^2(x)|
\end{aligned}$$

Since $\int_{B_R(0)} |\varphi_1^2(x) - \varphi_1^2(\frac{R_1}{R}x)|dx \to 0$ and $|B_{R_1}(0) - B_R(0)| \to 0$ as $R_1 \to R$ we have

$$\int_{B_R(0)} g(x)\hat{\varphi}^2(x)dx > 0.$$

So

$$\begin{aligned}
\lambda_1^+(R) &\leq \frac{\int_{B_R(0)} |\nabla \hat{\varphi}(x)|^2 dx}{\int_{B_R(0)} g(x)\hat{\varphi}^2(x) dx} \\
&= (R_1/R)^2 \frac{\int_{B_{R_1}(0)} |\nabla \varphi_1(y)|^2 dy}{\int_{B_{R_1}(0)} g(y){\varphi_1}^2(y) dy} \\
&= (R_1/R)^2 \lambda_1^+(R_1) \\
&< \lambda_1^+(R_1) + \epsilon.
\end{aligned}$$

The last inequality hold if we choose R_1 such that $R_1 > R$ and sufficiently close to R. If a such R_1 is chosen then for every $R' \in (R, R_1)$ we have

$$\lambda_1^+(R) - \lambda_1^+(R') < \lambda_1^+(R) - \lambda_1^+(R_1) < \epsilon.$$

Hence $\lambda_1^+(R)$ is a continuous function of R. $\Box$
Also by a quite similar argument we can prove the following theorem

Theorem 6. $R \to \lambda_1^-(R)$ is a continuous function of R.

Theorem 7. Let $\lambda_1^-(R) < \lambda < \lambda_1^+(R)$, then there exists $R' > R$ such that

$$\lambda_1^-(R') < \lambda < \lambda_1^+(R')$$

Proof. Let $\epsilon = \lambda_1^+(R) - \lambda$, so $\epsilon > 0$. By using the continuity of the function $R \to \lambda_1^+(R)$, there exists $R_1 > R$ such that $\lambda_1^+(R) - \lambda_1^+(R_1) < \epsilon$. Then we have

$$\lambda < \lambda_1^+(R_1).$$

Similarly with the continuity of the function $R \to \lambda_1^-(R)$, there exists $R_2 > R$ such that

$$\lambda > \lambda_1^-(R_2).$$

Now let $R' = \min\{R_1, R_2\}$, we have

$$\lambda_1^-(R') \le \lambda_1^-(R_2) < \lambda < \lambda_1^+(R_1) \le \lambda_1^+(R')$$

and so the proof is complete. □

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REMARKS ON UNCOMPLEMENTED SUBSPACES OF SOME OPERATOR SPACES

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ABSTRACT. We show that a closed subspace $S(E,F)$ of $L(E,F)$ in which weak operator topology and weak topology on sequences are coincide would be uncomplemented in both $L(E,F)$ and $W(E,F)$.

[1]

Let E and F be Banach spaces (real or complex). The symbols $L(E,F)$, $K(E,F)$ and $W(E,F)$ denote the Banach spaces of bounded operators (bounded linear maps), those are which compact, and those are which weakly compact. Many papers see ([**2**], [**7**], [**8**], and [**9**]) have been devoted to the question of uncomplementability of some subspaces of $L(E,F)$ in it.

In [**1**], we introduced and studied some operator spaces with the $\mathcal{K}$-property, i.e. subspaces of $L(E,F)$ in which weak operator topology and weak topology on sequences are coincide. It is well known

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that $K_{w^*}(E^*, F)$ the space of all compact weak*-weak continuous operators from E^* to F has the $\mathcal{K}$-property [**10**]. We reserve the symbol $S(E, F)$ for an arbitrary subspace of $L(E, F)$ containing $E^* \otimes F$ with the $\mathcal{K}$-property. The present note deals with the question of uncomplementability of $S(E, F)$ in $L(E, F)$ and $W(E, F)$. As a consequence we show $cca(\Sigma, E)$ is uncomplemented in $ca(\Sigma, E)$. All notions and terminologies used and not defined in this note can be found in [**4**] and [**5**]. Our purpose is to prove the following results.

THEOREM 1. *Suppose that F has a copy of ℓ_1 and E is not a Grothendieck space. Then $S(E^{**}, F)$ is uncomplemented in $L_{w^*}(E^{**}, F)$.*

THEOREM 2. *Suppose that E has a complemented copy of ℓ_1 and $\ell_\infty \otimes \ell_\infty \subseteq S(\ell_1, \ell_\infty)$. Then $S(E, \ell_\infty)$ is uncomplemented in $L(E, \ell_\infty)$.*

PROOF OF THEOREM 1. By the assumption there is an ℓ_1-basic sequence $(y_n)_n$ in F, i.e., there are $C_1,\ C_2 > 0$ such that

$$C_1 \sum_{n=1}^{\infty} \mid \eta_n \mid \leq \parallel \sum_{n=1}^{\infty} \eta_n y_n \parallel \leq C_2 \sum_{n=1}^{\infty} \mid \eta_n \mid, \qquad (\eta = (\eta_n)_n \in \ell_1).$$

Assume that $(x_n^*)_n$ is a normalized, weak* null but not weakly null sequence in E^*. There is $z^{**} \in B_{F^*}$ such that $z^{**} x_n^* \geq \epsilon$ $(n \in \mathbf{N})$. Define $\varphi : \ell_1 \to L(E^{**}, F)$ by $\varphi(\eta)x^{**} = \sum_{i=1}^{\infty} \eta_i x^{**}(x_i^*) y_i$, which is a linear map. φ is well defined since (y_n) is an ℓ_1-basic sequence. We now show that $\varphi(\eta)$ belongs to $L_{w^*}(E^{**}, F)$. To this aim, it will be enough to consider a weak*-null net $(x_\alpha^{**})_\alpha$ in $B_{E^{**}}$ and an element y^* of B_{F^*}, and proving that

$$lim_\alpha \mid \varphi(\eta)(x_\alpha^{**}) y^* \mid = 0. \tag{1}$$

Since $\sum \eta_n x_n^* y^*(y_n)$ is unconditionally converging in E, we have

$$lim_p sup_{x^{**}\in B_{E^{**}}} \mid \sum_{n=p+1}^{\infty} \eta_n x^{**}(x_n^*) y^*(y_n) \mid = 0. \tag{2}$$

Thanks to (2), given $\epsilon > 0$ for any $y^* \in B_{F^*}$ we can find $\overline{p} \in \mathbf{N}$ such that

$$sup_\alpha \mid \sum_{n=\overline{p}+1}^{\infty} \eta_n x_\alpha^{**}(x_n^*) y^*(y_n) \mid < \frac{\epsilon}{2}. \tag{3}$$

On the other hand

$$lim_\alpha \sum_{n=1}^{\overline{p}} \eta_n x_\alpha^{**}(x_n^*) y^*(f_n) = 0 \tag{4}$$

since $x_n^*(.)y_n \in L_{w^*}(E^{**}, F)$ for all $n \in \mathbf{N}$. The formulas (2) and (3) together give (1). Furthermore, using the closed graph Theorem we can prove easily that the linear map $\varphi : \ell_1 \to L_{w^*}(E^{**}, F)$ is continuous. Now assume on the contrary, there is a bounded projection $P : L_{w^*}(E^{**}, F) \to S(E^{**}, F)$. Boundedness of the sequence (y_n) shows that $P\varphi(e_n) = \varphi(e_n) = x_n^*(.)y_n$ is pointwise weakly null sequence, where e_n is the nth unit vector basis of ℓ_1. Therefore, it must be a weakly null sequence. By a theorem of Mazur ([**4**], page 4) there is a convex combination $S_n = \sum_{i=p_n+1}^{q_n} \alpha_{n,i}\varphi(e_i)$ of $(\varphi(e_n)_n)$ converging to zero in norm. Hence for enough large n, $\| S_n \| < C_1\epsilon$. On the other hand

$$C_1\epsilon < C_1 \sum_{p_n+1}^{q_n} \alpha_{n,i}(z^{**}x_i^*) \leq sup_{x^{**}\in B_{E^{**}}} \| \sum_{i=p_n+1}^{q_n} \alpha_{n,i} x^{**}(x_i^*) y_i \| \leq \| S_n \|$$

which is a contradiction. □

Theorem 1 has the following Corollaries.

Corollary 3. *Suppose E is not a Grothendieck space and F has a copy of ℓ_1. Then $K(E, F)$ is uncomplemented in $W(E, F)$.*

PROOF. Theorem 1 together with the two identifications $L_{w^*}(E^{**}, F) \simeq W(E, F)$, and $K_{w^*}(E^{**}, F) \simeq K(E, F)$ give the result. $\square$

COROLLARY 4. *Suppose E is not a Grothendieck space. Then $cca(\Sigma, E^*)$ is uncomplemented in $ca(\Sigma, E^*)$, where Σ is the σ-algebra of all Borel subsets of an infinite compact Hausdorff space Ω.*

PROOF. We recall that

$$cca(\Sigma, E^*) \simeq K_{w^*}(E^{**}, ca(\Sigma)), \qquad ca(\Sigma, E^*) \simeq L_{w^*}(E^{**}, ca(\Sigma)),$$

also $ca(\Sigma) \simeq C(\Omega)^*$ [**4**]. From [**3**], $C(\Omega)$ has a copy of c_0. Therefore, $ca(\Sigma)$ has a complemented copy of ℓ_1. Now, an appeal to Theorem 1 completes the proof. $\square$

Before we proceed to prove Theorem 2, we need the following lemma.

LEMMA 5. *Suppose $S(E, \ell_\infty)$ is complemented in $L(E, \ell_\infty)$ and $\varphi : \ell_\infty \to L(E, \ell_\infty)$ is an operator such that $\varphi(e_n) \in S(E, \ell_\infty)$ where e_n is the nth unit vector basis of c_0 in ℓ_∞. Then for any infinite subset M of* $\mathbf{N}$*, there is an infinite subset M_0 of M such that, $\varphi(\eta) \in S(E, \ell_\infty)$ for each $\eta \in \ell_\infty(M_0)$.*

PROOF. Let $\Gamma : L(E, \ell_\infty) \to S(E, \ell_\infty)$ be a bounded projection. Without loss of generality we can assume $M = \mathbf{N}$. It is easy to see that $\Gamma\varphi(e_n) = \varphi(e_n)$, $(n \in \mathbf{N})$. Hence by Proposition 5 of [**9**], there exists an infinite subset M_0 of $\mathbf{N}$ with $\varphi(\eta) \in S(E, \ell_\infty)$ for each $\eta \in \ell_\infty(M_0)$. $\square$

We are now in the position to prove Theorem 2.

PROOF OF THEOREM 2. Suppose $\Gamma : L(E, \ell_\infty) \to S(E, \ell_\infty)$ and $P : E \to H$ are bounded projections where H is a closed subspace of E isomorphic with ℓ_1. We define $\Delta : L(H, \ell_\infty) \to S(H, \ell_\infty)$ by $\Delta(T)h = \Gamma(TP)h$, $(h \in H)$ so Δ is a bounded projection. Therefore, we can assume $E = \ell_1$. Let $(f_n) \subseteq B_{\ell_\infty}$ and $z^* \in B_{\ell_\infty^*}$ with $z^*(f_n) \geq \epsilon$. Consider $\varphi : \ell_\infty \to L(\ell_1, \ell_\infty)$ defined by $\varphi(\eta)x = \sum \eta_n x_n f_n$, where $(x_n) \in \ell_1$ which is an operator. By Lemma 5, we can assume $\varphi(\eta)$ lies in $S(E, \ell_\infty)$ for each $\eta \in \ell_\infty$. Let $S_n = \sum_{i=p_n+1}^{q_n} \alpha_{n,i}\phi(e_i)$ be the same as in the proof of Theorem 1, by the same method for enough large n, $\| S_n \| < \epsilon$. Define $x = \frac{1}{q_n - p_n}\chi_{\{p_n+1,\ldots,q_n\}} \in \ell_1$ where $\chi_{\{p_n+1,\ldots,q_n\}}$ is the characteristic function on $\{p_n + 1, \ldots, q_n\}$. Then

$$\epsilon < (z^* \otimes x)(S_n) \leq \epsilon$$

which is a contradiction. □

Theorem 2 has the following Corollary.

Corollary 6. *If E^* has a copy of c_0, then $K(E, \ell_\infty)$ is uncomplemented in $L(E^{**}, \ell_\infty)$.*

Proof. The fact that in this case E^{**} has a complemented copy of ℓ_1 [**4**], and the identifications used in the proof of Corollary 3 completes the proof. □

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FUZZY SG-COMPACT SETS IN FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we define the concept of a fuzzy Sg-open, fuzzy Sg-closed and fuzzy Sg-Compact sets in fuzzy topological spaces. Then we state and prove some results.

[1]

1. Introduction

The concept of a fuzzy subset was first introduced by Zadeh in [2]. Since its inception, the theory of fuzzy subsets has developed in many directions and found applications in a wide variety of fields. The study of fuzzy, subsets and its application to various mathematical contexts has given rise to what is now commonly called fuzzy mathematics.

Fuzzy topological spaces (fts) is an important branch of fuzzy mathematics.

Fuzzy Topological Spaces first were introduced by C.L.chang in 1968 [3]. Up to now many researchers have been working on this field and developed it.

Sg-closed and Sg-open sets in general topology, were introduced for the

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key words: fuzzy topological spaces, fuzzy Sg-Open, fuzzy Sg-closed and fuzzy Sg-Compact.

first time by Bhattacharyya and Lahiri in 1987 [7]. In 1995, Sg-compact spaces were introduced independently by Caldas [5], and by Devi, Balachandra and Maki.
The purpose of this paper is:
(i) To contribute to he development of fts as introduced in [7].
(ii) To characterize and incerpret some properties by concepts peculiar to fuzzy Sg-closed and fuzzy Sg-Compact in fts only.
(iii) To manifest some departures between general topology (Sg-closed and Sg-Compact of topologies [4,7]) and fuzzy topology spaces(fts).
Therefore, in this paper We introduce the concept of Sg-Open and some general properties for Sg-Compact fts are obtained.

2. Preliminaries

Before we enter into the intended investigations properly, let us clarify some definitions, notations and results relevant to this paper, further details of which and other notations of the theory of fts can be found in references.
Let X be a spaces of points. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ from X to $[0,1]$.
Definition 1-1. Let A and B be fuzzy sets in X. Then

$$A = B \iff \mu_A(x) = \mu_B(x) \text{ for all } x \in X,$$
$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X,$$
$$C = A \cup B \iff \mu_C(x) = max\{\mu_A(x), \mu_B(x)\} \text{ for all } x \in X,$$
$$D = A \cap B \iff \mu_D(x) = min\{\mu_A(x), \mu_B(x)\} \text{ for all } x \in X,$$
$$E = A^c \iff \mu_E(x) = 1 - \mu_A(x) \text{ for all } x \in X.$$

more generally, for a family of fuzzy sets $\mathcal{A} = \{\mathcal{A}_\rangle \mid \rangle \in \mathcal{I}\}$, the Union $C = \cup_{i\in I} A_i$ and the intersection, $D = \cap_{i\in I} A_i$, are defined by

$$\mu_C(x) = Sup_{i\in I}\{\mu_{A_i}(x)\},\ x \in X,$$
$$\mu_D(x) = inf_{i\in I}\{\mu_{A_i}(x)\},\ x \in X.$$

The symbol Φ will be used to denote an empty, fuzzy set ($\mu_\phi(x) = 0$ for all $x \in X$).

For X, we have by definition $\mu_X(x) = 1$ for all x in X.

Definition 1-2. A fuzzy topology is a family τ of fuzzy sets in X which satisfies the following conditions:

(a) $\Phi, X \in \tau$

(b) if $A, B \in \tau$ then $A \cap B \in \tau$

(c)If $A_i \in \tau$ for each $i \in I$, then $\cup_I(A_i) \in \tau$.

τ is called a fuzzy topology for X, and the pair (X, τ) is a fuzzy topological space or fts for short.

Every member of τ is called a τ -open fuzzy set (or simply open fuzzy set). A fuzzy set is τ-closed or simply closed fuzzy set, if and only if its complement is τ-open.

Definition 1-3. Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$ is fuzzy set in X whose membership function is defined by

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \text{ for all } x \text{ in } X$$

Conversely, Let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f[A]$, is a fuzzy set in Y whose membership function is give by

$$\mu_{f[A]}(y) = \begin{cases} Sup_{z \in f^{-1}(y)}\{\mu_A(z)\}, & if\ f^{-1}[y]\ is\ not\ empty; \\ 0 & otherwise. \end{cases}$$

for all y in Y, where $f^{-1}[y] = \{x : f(x) = y\}$.

Definition 1-3. A fuzzy set U in a fts (X, τ) is a neighborhood of a fuzzy set A if and only if there is an open fuzzy set O such that $A \subseteq O \subseteq U$. Where as for two fuzzy sets A and B, $A \leq B$ means $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$.

Definition 1-4. A fuzzy point in a set X is a fuzzy set $p : X \to I$, I

being the closed unit interval in X, such that

$$p(x) = \begin{cases} t & for\ x = x_p; \\ 0 & otherwise. \end{cases}$$

where $t \in (0,1), x_p$ is called the support p and $t,$ its value.
Also, a fuzzy point p is said to belong to a fuzzy set A in X (notation: $p \in A$) if and only if $p(x_p) \leq \mu_A(x_p)$. If A is a subset of X, we shall denote the characteristic function of A, also by A.

Definition 1-5. Let A and B be fuzzy sets in a fts (X, τ), and let $B \leq A$ then B is called an interior fuzzy set of A if and only if A is a neighborhood of B the interior of A and is denoted by A^0. The closure and interior of a fuzzy set A in a fts (X, τ) are defined respectively by:

$$\overline{A} = inf\{D : D \geq A, D \in \tau\},$$
$$A^0 = sup\{\text{D:D}\leq\text{A,D}\in \tau\}.$$

It is easily seen the $\overline{A}$ is the smallest closed fuzzy set larger than A and A^0 is the largest open fuzzy set smaller than A.

Definition 1-7. A family V of fuzzy sets is a cover of a fuzzy set A if and only if $B \leq \cup\{v : v \in V\}$. It is an open cover if and only if each member of V is an open fuzzy set. A subcover of V is a subfamily of V which is also a cover.

Definition 1-8.A fts (X, τ) is compact if and only if each open cover has a finite subcover.

Definition 1-9. If τ is fts on a set X and $Y \subseteq X$, then $\tau \mid_Y$ denotes the restriction of τ to Y.

$$\tau \mid_Y = \{Y \cap U : U \in \tau\} = \{\mu \mid_Y : \mu \in \tau\}$$

The closure operator in a space is denoted by [0]. When we wish to underscore that the closure is taken in a space (X, τ), we write $[0]_{(X,\tau)}$ instead of [0]. Fuzzy points with different supports will be called distinct.

Srivastava [9] defined a "fuzzy T_1 -topological space" as follows.

Definition 1-10. An fts (X, τ) is fuzzy T_1 if and only if for any two distinct fuzzy points p, q in $X, \exists U, V \in \tau$ such that $p \in U, q \notin U, q \in V, p \notin V$.

3. MAIN DEFINITIONS:

Definition 2-1. A fuzzy set A in an fts X is said to be:

(i) A fuzzy semiopen set of X if $A \leq cl(IntA)$,

(ii) A fuzzy α-open set of X if $A \leq Int(cl(IntA))$,

(iii) A fuzzy preopen set of X if $A \leq Int(cl(A))$.

The family of all fuzzy semiopen (resp. fuzzy α-open, fuzzy preopen) sets of a fts X is denoted by $FSO(X)$, (resp. $F\alpha(X)$, $FPO(X)$).

Definition 2-2. A fuzzy set A is called fuzzy semiclosed (or $FSC(X)$) (resp. fuzzy α-closed, fuzzy preclosed) if $A^c \in FSO(X)$ (resp. $F\alpha(X)$, $FPO(X)$).

Remark. Every fuzzy open set is fuzzy α-open and every fuzzy α-open set is fuzzy semiopen as well as fuzzy peropen, but the separate converses need not be true.

Definition 2-3.Let A be is a fuzzy set in a fts X, the fuzzy semi-interior of A, denoted by $FSInt(A)$, the fuzzy semi-closure of A, denoted by $FSCL(A)$ and the fuzzy semi-kernel of A denoted by Fsker(A) and are defined as follows:

$$FSInt(A) = \vee\{U : U \in FSO(X) \text{ and } U \leq A\}.$$
$$FSCL(A) = \wedge\{U : U \in FSCL(X) \text{ and } A \leq U\}.$$
$$FSKer(A) = \wedge\{U : U \in FSO(X) \text{ and } A \leq U\}.$$

It is well known that $FSIn(A) = A \wedge cl(IntA)$ and $FSCL(A) = A \vee Int(clA)$.

Recall that a fuzzy set A of a fts X, called fuzzy $Sg{-}open$ (or, FSGO(X)) is every fuzzy semiclosed subset of A included in the fuzzy semi-interior of A.

Definition 2-4. A mapping $f : X \to Y$ is called fuzzy Sg-continuous, (or fuzzy Sg-irresolute) if $f^{-1}(u) \in$ FSGO(X) (or FSGC(X)) for every open set (or fuzzy Sg-closed) u of Y.

Theorem 2-5.

I) If A is fuzzy Sg-closed and B is fuzzy closed set Then $A \vee B$ is also fuzzy Sg-closed.

II) The intersection of a fuzzy Sg-open and an fuzzy open set is always fuzzy Sg-open.

III) The union of a fuzzy Sg-closed set and a fuzzy semi-closed set need not be fuzzy Sg-closed set, in particular even finite union of fuzzy Sg-closed sets need not be fuzzy Sg-closed set.

Proof. Assertions (I), (II) and (III) are obvious.

We prove that arbitrary intersection of fuzzy Sg-closed sets is fuzzy Sg-closed.

In order to do that, we need first the following lemma.

Lemma 2-6. Let (X, τ) be a fts, then:

(i) A fuzzy subset A of X is fuzzy Sg-closed if and only if fscl $(A) \leq$ fsker (A).

(ii) Every singleton $\{x_p\}$ is either nowhere dense or fuzzy preopen.

Theorem 2-7. Arbitrary intersection of fuzzy Sg-closed sets is a fuzzy Sg-closed set.

proof: Let $\{A_i : i \in I\}$ be an fts (X, τ) and let $A = \bigwedge_{i \in I} A_i$. Let $x_p \in fsc(A)$.

In the notion of Lemma 2-6 (ii), we consider the following tow cases:

Case 1. $\{x_p\}$ is nowhere dense. If $x \notin A$, then for some $j \in I$we have $x_p \notin A_j$. Since nowhere dense sets are fuzzy semi-closed, then

$x_p \notin fsker(A_j)$. On the other hand, by Lemma 2-6 (i), $x_p \in fscl(A) \leq fscl(A_j) \leq fsker(A_j)$, since A_j is fuzzy Sg-closed. By contradiction, $x_p \in A$ and hence $x_p \in fsker(A)$.

Case 2. $\{x_p\}$ is fuzzy preopen. Set $F = int(cl\{x_p\})$.
Assume that $x \notin fsker(A)$. Then there exists a fuzzy semi-closed set S containing x_p such that $S \bigwedge A = \phi$. Now, $x_p \in F = int(cl\{x_p\}) \leq int(cl(S)) \leq S$,
Since S is fuzzy semi-closed. Since F is fuzzy semiopen set containing x_p and since $x_p \in fscl(A)$, then $F \bigwedge A \neq \phi$. Since $F \leq S$, then $S \bigwedge A \neq \phi$.
By contradiction, $x_p \in fsker(A)$.
Thus, in both cases $x_p \in fsker(A)$. By Lemma 2-6 (i), A is fuzzy Sg-closed.
Corollary 2-8. (i) Any Union of fuzzy Sg-open is a fuzzy Sg-open set.
(ii) A fuzzy subset A of a fts (X, τ) is fuzzy Sg-closed if and only if A is intersection of fuzzy Sg-closed sets.
If $B \leq A$ and A is fuzzy open and fuzzy Sg-closed, then B is fuzzy Sg-closed in the fuzzy subspace A if and only if B is fuzzy Sg-closed in X.
Since a fuzzy subset is fuzzy regular open if and only if is fuzzy α-open and fuzzy Sg-closed.
We obtain the following result:
Proposition 2-9. Let B be a fuzzy regular open subset of a fts (X, τ). If $A \leq B$ and A is fuzzy Sg-open in $(B, \tau|_B)$, then A is fuzzy Sg-open in X.
Definition 2-10. let X be a fts and A be a fuzzy set in X. A collection C of fuzzy sets in X is said to be a fuzzy cover of A if and only if $[\bigvee\{c : c \in C\}](x) = 1$, for all $x \in A$. If the members of C are fuzzy open (resp, fuzzy semiopen, fuzzy Sg-open), C is called a fuzzy open (resp,

fuzzy semiopen, fuzzy Sg-open) cover of A.

A fuzzy open (resp, fuzzy semiopen, fuzzy Sg-open) cover C of a fuzzy set A in X said to have a finite subcover C_0 for A if and only if there exists a finite subcollection $C_0 = \{c_1, c_2, ..., c_n\}$ (say) of C such that $[\bigvee_{i=1}^{n} \{c_i\}] \geq A$. A fuzzy set A in X is fuzzy compact (resp, fuzzy semicompact, fuzzy Sg-compact) if and only if every fuzzy open (resp, fuzzy semiopen, fuzzy Sg-open) cover of A has a finite subcover for A.

Definition 2-11. A fuzzy subset B of a fts X is said to be fuzzy Sg-compact if B is fuzzy Sg-compact as a fuzzy subspace of X.

Definition 2-12. A fuzzy subset B of a fts X is said to be fuzzy Sg-compact relative to X if, for every collection $\{c_i \mid i \in J\}$ of Sg-open subsets of X such that $B \leq \{c_i : i \in J\}$, there exists a finite subset J_0 of J such that $B \leq \{c_i : i \in J_0\}$.

Theorem 2-13. Every fuzzy Sg-closed subset of a fuzzy Sg-compact space X is fuzzy Sg-compact relative to X.

proof. Let A be fuzzy Sg-closed subset of X, then A^c is fuzzy Sg-open in X. Let $M = \{G_i : i \in J\}$ be a cover of A by fuzzy Sg-open subsets in X, then $M^* = M \vee A^c$ is a fuzzy Sg-open cover of X, i.e., $X = (\vee\{G_i : i \in J\}) \vee A^c$. By hypothesis, X is fuzzy Sg-compact, hence M^* is reducible to finite cover of X, say $X = G_{i_1} \vee G_{i_2} \vee ... \vee G_{i_m} \vee A^c$, $G_{i_k} \in M$. But A and A^c are disjoint, hence A $G_{i_1} \vee G_{i_2} \vee ... \vee G_{i_m}$, $G_{i_k} \in M$. We have just shown that any fuzzy Sg-open cover M of A contains a finite subcover, i.e., A is fuzzy Sg-compact relative to X.

Theorem 2-14.

i) A fuzzy Sg-continuous image of a fuzzy Sg-compact space is fuzzy compact.

ii) If a map $f : X \longrightarrow Y$ is fuzzy Sg-irresolute and a fuzzy subset B of X is fuzzy Sg-compact relative to X, the image $f(B)$ is fuzzy Sg-compact relative to Y.

Proof.

i) Let $f : X \longrightarrow Y$ be a fuzzy Sg-continuous map from a fuzzy Sg-compact space X onto a fts Y. Let $\{A_i : i \in J\}$ be an fuzzy open cover of Y. Then $\{f^{-1}(A_i) : i \in J\}$ is a fuzzy Sg-open cover of X. Since X is fuzzy Sg-compact, it has a finite fuzzy subcover, say $\{f^{-1}(A_1), ..., f^{-1}(A_n)\}$. Since f is onto $\{A_1, ..., A_n\}$ is a fuzzy cover of Y and so Y is fuzzy compact.

ii) Let $\{A_i : i \in J\}$ be any collection of fuzzy Sg-open fuzzy subsets of Y such that $f(B) \leq \vee\{A_i : i \in J\}$, then

$B \leq \vee\{f^{-1}(A_i) : i \in J\}$ holds.

By hypothesis there exists a finite fuzzy subset J_0 of J such that

$B \leq \vee\{f^{-1}(A_i) : i \in J_0\}$. Therefore, we have $f(B) \leq \vee\{A_i : i \in J_0\}$ which shows that $f(B)$ is fuzzy Sg-compact relative to Y.

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A REMARK ON CYCLIC VECTORS OF BERGMAN, HARDY AND DIRICHLET SPACES OF FINITELY CONNECTED DOMAINS

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ABSTRACT. Let $\mathbf{G}$ be a finitely connected domain in the complex plane $\mathbf{C}$, and $K_1, \cdots K_N$ be the bounded components of $\mathbf{C}\backslash G$ such that at least one K_i has a nonempty interior.

In this note, we show that if $f \in H^\infty(G_0)$, where $G_0 = G \cup K_1 \cup \cdots \cup K_N$ then f is not a cyclic vector for $H^\infty(G_0)$ in the weighted Bergman space $L^p_a(G, wdm), 1 \leq p < \infty$, where w is a positive continuous function in $L^1(G)$, the Hardy space $H^p(G), (1 \leq p < \infty)$, and the Dirichlet space $D(G)$. In particular, in this case, the polynomials are not dense in $L^p_a(G, wdm), H^p(G), (1 \leq p < \infty)$, and $D(G)$.

1

1. INTRODUCTION

Let G be a domain in the complex plane $\mathbf{C}$. An analytic function f in G belongs to the weighted Bergman space $L^p_a(G, wdm), 1 \leq p < \infty$ if $\int_G |f|^p wdm < \infty$, where m is the area measure on $\mathbf{C}$. In [5], it is shown

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that if w is a positive continuous function in $L^1(G)$ then $L^p_a(G, wdm)(1 \leq p < \infty)$ is a Banach space.

The Hardy space $H^p(G)$ consists of all analytic functions f defined on G such that there is a harmonic function $u : G \to [0, +\infty)$ with $|f|^p \leq u$ on G. It is well known that $H^p(G)(1 \leq p < \infty)$ is a Banach space. Also, the Dirichlet space $D(G)$ is the Hilbert space of functions f analytic in G whose derivative f' lies in $L^2_a(G, dm)$, [9, 10].

Let $H^\infty(G)$ denote the space of bounded analytic functions on G with the supremum norm. Clearly $H^\infty(G)$ is a subset of $L^p_a(G, wdm)$ and $H^p(G)$. Suppose G is a finitely connected domain in the complex plane $\mathbf{C}$ and $K_1, \cdots, K_N$ are the bounded components of $\mathbf{C}\backslash G$. Put $G_0 = G \cup K_1 \cup \cdots \cup K_N$. A function f in $L^p_a(G, wdm)$ (or $H^p(G)$) is cyclic for $H^\infty(G_0)$, if the vector subspace $\{\varphi f : \varphi \in H^\infty(G_0)\}$ is dense in $L^p_a(G, wdm)$ (or $H^p(G)$). Also a function f in $D(G)$ is cyclic for $H^\infty(G_0) \cap D(G)$ if the vector subspace $\{\varphi f : \varphi \in H^\infty(G_0) \cap D(G)\} \cap D(G)$ is dense in $D(G)$.

Main Result

Theorem. Suppose G is a finitely connected domain in the complex plane $\mathbf{C}$ and $K_1, \cdots, K_N$ are the bounded components of $\mathbf{C}\backslash G$ such that at least one K_i has a nonempty interior. If $f \in H^\infty(G_0)$ then f is not a cyclic vector for $H^\infty(G_0)$ in the weighted Bergman space $L^p_a(G, wdm), 1 \leq p < \infty$, where w is a positive continuous function in $L^1(G)$, the Hardy space $H^p(G), 1 \leq p < \infty$, and the Dirichlet space $D(G)$.

Proof. We only prove the theorem for the weighted Bergman space $L^p_a(G, wdm)$. The proofs for the Hardy and Dirichlet spaces are similar.

On the contrary, suppose $f \in H^\infty(G_0)$ is a cyclic vector for $H^\infty(G_0)$ in $L^p_a(G, wdm)$. Let $\varphi \in H^\infty(G) \subseteq L^p_a(G, wdm)$ [for Dirichlet space let

$\varphi \in H^\infty(G) \cap D(G)$]. So there exists a sequence $\{\varphi_n\}$ in $H^\infty(G_0)$ such that $\varphi_n f \to \varphi$ in $L_a^p(G, wdm)$. Therefore, $\varphi_n f$ converges uniformly on compact subsets of G [5, lemma 2]. We can choose a closed rectifiable curve γ in G such that $K_i \subset Int\gamma, i = 1, \cdots N$, where Int$\gamma$ is the interior points of γ. Now, by Cauchy's Integral formula,

$$|\varphi_n(z)f(z) - \varphi_m(z)f(z)| \leq \int_\gamma |\frac{\varphi_n(w)f(w) - \varphi_m(w)f(w)}{w - z}||dw|$$

$$z \in K_1 \cup \ldots \cup K_N.$$

Since $\{\varphi_n f\}$ converges uniformly on γ, it is uniformly Cauchy on $K_1 \cup \ldots \cup K_n$. Therefore, it is uniformly convergent on compact subsets of G_0. Let ψ be the limit of $\{\varphi_n f\}$. In fact, ψ is the analytic extension of φ on G_0. By the maximum modulus Theorem,

$$\sup_{z \in G_0} |\psi(z)| = \sup_{z \in G} |\varphi(z)| < \infty.$$

Without loss of generality, suppose IntK_1 is nonempty. Choose z_1 in $IntK_1$ and put $g(z) = \frac{1}{z - z_1}$. It is obvious that $g(z) \in H^\infty(G)$, but does not have any analytic extension on G_0. This contradiction shows that f is not a cyclic vector for $L_a^p(G, wdm)$.□

Corollary 1. Suppose G is a finitely connected domain in the complex plane $\mathbf{C}$ and $K_1, \cdots, K_N$ are the bounded components of $\mathbf{C} \backslash G$ such that at least one K_i has a nonempty interior. If $G_0 = G \cup K_1 \cup \cdots \cup K_N$, then $H^\infty(G_0)$ is not dense in the weighted Bergman space $L_a^p(G, wdm), (1 \leq p < \infty)$, where w is a positive continuous function in $L^1(G)$, and the Hardy space $H^p(G)(1 \leq p < \infty)$. Also $H^\infty(G_0) \cap D(G)$ is not dense in the Dirichlet space $D(G)$.

Proof. Put $f \equiv 1$, the constant function, in the theorem. □

Remark. It follows from [10, Proposition 12] that $H^\infty(G) \cap D(G)$ is dense in $D(G)$.

Density of polynomials in $H^2(G)$ for crescents G have been investigated in [1,2,3]. Also [4] contains results about cyclic vectors for the Hardy operators. Moreover it is an old problem that for which w and for which G, the polynomials or rational functions are dense in $L^p_a(G, wdm)$ or not [6, 7, 8, 11, 12]. The following corollary, follows immediately from the theorem.

Corollary 2. Suppose G is a finitely connected domain in the complex plane $\mathbf{C}$ and $K_1, \cdots, K_N$ are the bounded components of $\mathbf{C}\backslash G$ such that at least one K_i has a nonempty interior. Then the polynomials are not dense in the weighted Bergman space $L^p_a(G, wdm), (1 \leq p < \infty)$, where w is a positive continuous function in $L^1(G)$, the Hardy space $H^p(G), (1 \leq p < \infty)$, and the Dirichlet space $D(G)$. □

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A SHORT PROOF FOR THE EXISTENCE OF HAAR MEASURE ON COMMUTATIVE HYPERGROUPS

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ABSTRACT. In this short note, we have given a short proof for the existence of the Haar measure on commutative locally compact hypergroups based on functional analysis methods by using Markov-Kakutani fixed point theorem.

1. INTRODUCTION

A fundamental open question about hypergroups is the existence of Haar measure for any hypergroup. If a hypergroup K is compact or discrete, then K possesses a Haar measure. All known examples have a Haar measure [6, §5]. Spector in [11] claims that any commutative hypergroup possesses a Haar measure but as Ross in [9] mentioned there are several technical problems in his proof. Ross in [9] has given a lengthy proof for existence of Haar measure on commutative hypergroups. Recently Izzo in [5] has given a short proof of the existence of Haar measure on a commutative locally compact group by using the Markov-Kakutani fixed-point theorem [1, pp. 155-156]. Based on his idea, we give a short proof of the existence of Haar measure on commutative hypergroups.

For the reader's convenience, we include the Markov-Kakutani fixed point theorem. Let $\mathcal{S}$ be a compact convex subset of a Hausdorff topological vector space and $\mathcal{F}$ be a commutative family of continuous affine mappings of $\mathcal{S}$ into $\mathcal{S}$. Then there exists $p \in \mathcal{S}$ such that $\Lambda(p) = p$ for all $\Lambda \in \mathcal{F}$ (for a proof see [1]).

Note 1.1. For a vector space X, let $X^{\#}$ be the space of all linear functionals on X with the weak topology induced by X. Then, if C is a closed subset of $X^{\#}$ such that the set $\{\Lambda x : \Lambda \in C\}$ is bounded, for any $x \in X$, then C is compact (see [3, PP. 423-424]).

Lemma 1.2. *Let K be a hypergroup and U a symmetric neighborhood of the identity $e \in K$. Then there exists a subset M of K such that for any finite subset $\{g_1, g_2, \cdots g_n\}$ of K, the set $g_1 * g_2 * \cdots * g_n * U * U$ contains at least one element of M and the set $g_1 * g_2 * \cdots * g_n * U$ contains at most one element of M.*

Proof: Let

$$\begin{aligned}\mathcal{A} = \{T \subseteq K : \text{ for any } \quad p \neq q \in T, \text{there is a finite subset} \{g_1, g_2, \cdots, g_n\} \\ \text{of} \quad K \quad \text{such that} \quad p \notin q * A * \breve{A}, \quad \text{where} \quad \breve{A} = U * \breve{g_n} * \cdots * \breve{g_1}\}.\end{aligned}$$

Then $\mathcal{A}$ is non-empty and any chain $\{T_\alpha\}_{\alpha \in I}$ in $\mathcal{A}$ has an upper bound $\cup_{\alpha \in I} T_\alpha$. So by Zorn's Lemma $\mathcal{A}$ has a maximal element M. By using [6, 4.1A, 4.1B], we have $M \cap g * U * U \neq \emptyset$. Now for $\{g_1, g_2, \cdots, g_n\}$ an arbitrary finite subset of K, we have

$$\begin{aligned}M \cap g_1 * g_2 * \cdots * g_n * U * U = M \cap (\cup_{x \in g_1 * g_2 * \cdots * g_n} x * U * U) = \\ \cup_{x \in g_1 * g_2 * \cdots * g_n} (M \cap x * U * U) \neq \emptyset.\end{aligned}$$

To show that M intersects $g_1, g_2, \cdots, g_n * U$ at most at one point, let there are s_1 and s_2 in M that $s_1 \neq s_2$ and $s_i \in g_1 * g_2 * \cdots * g_n * U$ for $i = 1, 2$. Then by using [6, 4.1A, 4.1B] we have $s_1 \in s_2 * A * \breve{A}$, where A is $U * \breve{g_n} * \cdots * \breve{g_2}$ and this contradicts $M \in \mathcal{A}$. So the proof of the

Lemma is complete. $\square$

Theorem 1.3. *Every commutative hypergroup K has a left Haar measure.*

Proof: Let $C_{00}(K)^{\#}$ be the space of all linear functionals on $C_{00}(K)$. We consider on $C_{00}(K)^{\#}$ the weak topology generated by $C_{00}(K)$. It is clear that if there exists a $\Lambda \in C_{00}(K)^{\#}$ such that $f(\Lambda) = 0$ for all $f \in C_{00}(K)$, then $\Lambda = 0$. So $C_{00}(K)^{\#}$ with this topology is a locally convex space (see[4, P. 50]) . Let U be a fixed symmetric neighborhood of the identity $e \in K$ with compact closure. Let $\mathcal{S}$ be the set of all positive linear functionals Λ on $C_{00}(K)$ that satisfy the following two conditions:

(i) $\Lambda(f) \leq 1$ whenever $f \leq 1$ in $C_{00}^{+}(K)$ and $sptf \subseteq a_1 * a_2 * \cdots * a_r * U$ for some finite subset $\{a_1, a_2, \cdots, a_r\}$ in K,

(ii) $\Lambda(f) \geq 1$ whenever $f \leq 1$ in $C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_r * U * U$ for some finite subset $\{a_1, a_2, \cdots, a_r\}$ in K.

Then one can easily check that $\mathcal{S}$ is closed and convex. Moreover, any $f \in C_{00}^{+}(K)$ can be written as a finite sum of non-negative continuous functions, each of which has support in $a * U$ for some $a \in K$. To see this, let $sptf = C$, (compact set). Then $C \subseteq \cup_{1 \leq i \leq n} a_i * U$ for some $a_i \in K$, $1 \leq i \leq n$. By the partition of unity on compact sets, there are $h_i \in C_{00}^{+}(K)$ such that $0 < \frac{h_i}{f} \leq 1$ on C. That is for any $x \in C$, $0 < h_i(x) \leq f(x)$ and $h_1(x) + h_2(x) + \cdots + h_n(x) = f(x)$. Now it follows from (i) that the set $\{\Lambda(f) : \Lambda \in \mathcal{S}\}$ is bounded. So by Note 1.1, $\mathcal{S}$ is compact.

To see $\mathcal{S}$ is non-empty, let M be as in Lemma 1. Put $\Lambda(f) = \sum_{s \in M} f(s)$, then $\Lambda \in \mathcal{S}$. Indeed, if $f \in C_{00}^{+}(K)$ and $f \leq 1$ with $sptf \subseteq a_1 * a_2 * \cdots * a_n * U$ for some $a_i \in K$, $1 \leq i \leq n$, then by Lemma 1, M intersects

$a_1 * a_2 * \cdots * a_n * U$ at most at one point. Hence $\Lambda(f) \leq 1$. If $f \in C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_n * U * U$ for some $a_i \in K, \quad 1 \leq i \leq n$, then again by Lemma 1, M intersects $a_1 * a_2 * \cdots * a_n * U * U$ at least at one point. So $\Lambda(f) \geq 1$.

For each $x \in K$, let $T_x : C_{00}(K)^{\#} \to C_{00}(K)^{\#}$ is defined by $T_x\Lambda(f) = \Lambda({}_xf)$ for $f \in C_{00}(K)$. Then it is easy to see that T_x is affine and $T_x(\mathcal{S}) \subseteq \mathcal{S}$. Indeed, let $\Lambda \in \mathcal{S}$. If $f \in C_{00}^{+}(K)$ and $f \leq 1$ with $sptf \subseteq a_1 * a_2 * \cdots * a_n * U$ for some $a_i \in K, \quad 1 \leq i \leq n$, then ${}_xf \in C_{00}^{+}(K)$ (see [6, 4.2E]) and ${}_xf \leq 1$ with $spt({}_xf) \subseteq \breve{x} * a_1 * a_2 * \cdots * a_n * U$. So by (i) $\quad \Lambda({}_xf) \leq 1$. If $f \in C_{00}^{+}(K)$ and $f = 1$ on $a_1 * a_2 * \cdots * a_n * U * U$ for some $a_i \in K, \quad 1 \leq i \leq n$, then ${}_xf \in C_{00}^{+}(K)$ and ${}_xf = 1$ on $\breve{x} * a_1 * a_2 * \cdots * a_n * U * U$. So by (ii), $\quad \Lambda({}_xf) \geq 1$.

Also T_x is continuous, since if $\lim_\alpha \Lambda_\alpha = \Lambda$ in $\mathcal{S}$, then for any $f \in C_{00}(K)$,

$$\lim_{\alpha} |T_x\Lambda_\alpha(f) - T_x\Lambda(f)| = \lim_{\alpha} |\Lambda_\alpha({}_xf) - \Lambda({}_xf)| = 0.$$

Moreover for $x, y \in K$,

$$T_x(T_y\Lambda) = T_{x*y}\,\Lambda = T_{y*x}\,\Lambda = T_y(T_x\Lambda)$$

for any $\Lambda \in C_{00}(K)^{\#}$. This shows that the family $\mathcal{F} = \{T_x : x \in K\}$ and $\mathcal{S}$ (as above) have all properties in Markov-Kakutani fixed-point theorem. So there exists $\Lambda_0 \in \mathcal{S}$ such that $T_x\,\Lambda_0 = \Lambda_0$ for all $x \in K$. In another words

$$T_x(\Lambda_0 f) = \Lambda_0({}_xf) = \Lambda_0(f) \qquad \text{for all} \quad a \in K \quad \text{and} \quad f \in C_{00}(K).$$

Now since all elements of $\mathcal{S}$ are non-zero positive linear functionals on $C_{00}(K)$, by [6, §5.2] the proof is complete. $\square$

Remark 1.4. Can the above proof be modified to show that every amenable hypergroup has a left Haar measure, using Day's generalization of Markov-Kakutani fixed-point theorem [2, Theorem 1] (see also [7, Theorem 4.2])?
(For an extension to hypergroups see [10, Theorem 3.3.1].)
It is attempted such modification, but there is a problem in the continuity of action of hypergroup K on $\mathcal{S}$ (Page 33) defined by

$$(x, \Lambda) \longmapsto T_x\Lambda \quad \text{where} \quad T_x\Lambda(f) = \Lambda({}_xf) \quad \text{for} \quad f \in C_{00}(K) \quad \text{(Page 34)}.$$

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AMENABILITY AND WEAK AMENABILITY ON THE SECOND DUAL OF A BANACH ALGEBRAS WITH TWO ARENS MULTIPLICATIONS

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Abstract. Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}^{**}$ be the second dual of $\mathfrak{A}$. Tow Arens multiplications on $\mathfrak{A}^{**}$ are indicated by $(\mathfrak{A}^{**}, \Box)$ and $(\mathfrak{A}^{*}, \Diamond)$. In this paper, we study amenability and weak amenability on the second dual of $\mathfrak{A}^{**}$ with two Arens multiplications. First, we investigate the links between amenability and Arens regularity such that amenability of $\mathfrak{A}^{**}$ implies the Arens regularity of $\mathfrak{A}$. Also, we study, with some conditions on a Banach algebra of $\mathfrak{A}$, which ensure that $(\mathfrak{A}^{**}, \Box)$ is amenable (weakly amenable) if and only if $(\mathfrak{A}^{**}, \Diamond)$ is amenable (weakly amenable).

[1]

1. Introduction.

Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$-bimodule. Thus there are bilinear maps $(a,x) \longrightarrow a.x$ and $(a,x) \longrightarrow x.a$ from $\mathfrak{A} \times X$ into X such that, for $a, b \in \mathfrak{A}$, $x \in X$, $(ab).x) = a.(b.x)$, $x.(ab) = (x.a).b$, $a.(x.b) = (a.x).b$ and

$$\|a.x\| \leq \|a\|\|x\|, \quad \|x.a\| \leq \|x\|\|a\|.$$

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If X is a Banach $\mathfrak{A}$-bimodule, then the dual space X^* is a Banach $\mathfrak{A}$-bimodule with the actions defined by the following way:

For a in $\mathfrak{A}, x$ in X, and x^* in X^*,

$$< a.x^*, x >=< x^*, x.a >, < x^*.a, x >=< x^*, a.x > .$$

Let $\mathfrak{A}^*$ be the dual space of $\mathfrak{A}$. For $f \in \mathfrak{A}^*$, $a \in \mathfrak{A}$, we denote fa and af the elements of $\mathfrak{A}^*$ by

$$< fa, b >=< f, ab >, < af, b >=< f, ba >$$

where, $<, >$ is used the dual pairing between elements of $\mathfrak{A}^*$ and $\mathfrak{A}$. A derivation into an $\mathfrak{A}$-bimodule X is a linear map $D; \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = a.D(b) + D(a).b \quad (a, b \in \mathfrak{A}).$$

If $x \in X$, define

$$\delta_x(a) = a.x - x.a.$$

Then δ_x is a derivations into X, such derivations are called inner. The Banach algebra $\mathfrak{A}$ is amenable if, for every Banach $\mathfrak{A}$-bimodule X, every continuous derivation $D; \mathfrak{A} \longrightarrow X^*$ is inner. If $X^* = \mathfrak{A}^*$, we say that $\mathfrak{A}$ is weakly amenable. See ([6], Section 5).

If $\mathfrak{A}$ has a bounded approximate identity then $\mathfrak{A}^*\mathfrak{A}$ and $\mathfrak{A}\mathfrak{A}^*$ are closed linear subspace of $\mathfrak{A}^*$. As is well-known [1], the second dual $\mathfrak{A}^{**}$ of $\mathfrak{A}$ endowed with the either Arens multiplications is a Banach algebra. The Arens multiplications can be determined in the following way.

For $m \in \mathfrak{A}^{**}$ and $n \in \mathfrak{A}^{**}$, if we regard $\mathfrak{A}$ as a subspace of the second dual $\mathfrak{A}^{**}$, we can find bounded nets (m_α) and (n_β) in $\mathfrak{A}$ with $\hat{m}_\alpha \longrightarrow m$ and $\hat{n}_\beta \longrightarrow n$, in the weak* topology $\sigma(\mathfrak{A}^{**}, \mathfrak{A}^*)$ [$\hat{\mathfrak{A}}$ is image of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ under the canonical mapping]. So, the first Arens multiplications indicated by $m\Box n$ is given by

$$m\Box n = w^* - \lim_{\alpha} w^* - \lim_{\beta} \hat{m}_\alpha \hat{n}_\beta.$$

The second Arens multiplication indicated by $m\Diamond n$ is given by

$$m\Diamond n = w^* - \lim_{\beta} w^* - \lim_{\alpha} \hat{m}_\alpha \hat{n}_\beta.$$

Thus, the order in which the limits are taken distinguishes between the multiplications. Moreover, the first (resp. second) Arens multiplication is characterized by the two properties:

(i) for each $n \in \mathfrak{A}^{**}$, the mapping $m \longrightarrow m\Box n$ (resp. $m \longrightarrow n\Diamond m$) is weak*-weak* continuous on $\mathfrak{A}^{**}$.

(ii) for each $a \in \mathfrak{A}$, the mapping $m \longrightarrow \hat{a}\Box m$ (resp. $m \longrightarrow m\Diamond \hat{a}$) is weak*-weak* continuous on $\mathfrak{A}^{**}$.

However, for certain $n \in \mathfrak{A}^{**}$, the mapping $m \longrightarrow n\Box m$ (resp. $m \longrightarrow m\Diamond n$) is, in general, not weak*-weak* continuous on $\mathfrak{A}^{**}$. Whence the *topological center of* $\mathfrak{A}^{**}$ with respect to *first* and *second* left and right Arens multiplication are defined by

$$Z_1 = \{m \in \mathfrak{A}^{**} : n \longrightarrow m\Box n \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

$$Z_2 = \{m \in \mathfrak{A}^{**} : n \longrightarrow n\Diamond m \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

It is clear that $\hat{\mathfrak{A}} \subseteq Z_1 \cap Z_2$ and that Z_i $(i = 1, 2)$ is closed subalgebra of $\mathfrak{A}^{**}$.

The algebra $\mathfrak{A}$ is said to be *Arens regular* if, for each n and m in $\mathfrak{A}^{**}$, $n\Box m = n\Diamond m$. In this case $Z_1 = Z_2 = \mathfrak{A}^{**}$.

The following lemma follows easily from the definitions.

Lemma 1. *Let Z_1 and Z_2 be the left and right topological centers of $\mathfrak{A}^{**}$. Then*

(i) $m \in Z_1$ *if and only if* $m\Box n = m\Diamond n$ *for all* $n \in \mathfrak{A}^{**}$.

(ii) $m \in Z_2$ *if and only if* $n\Box m = n\Diamond m$ *for all* $n \in \mathfrak{A}^{**}$.

Proof. We shall prove only the assertion (i), the proof of the assertion (ii) is similar. For all $n \in \mathfrak{A}^{**}$, we can find a net (n_α) in $\mathfrak{A}$ such that $w^* - \lim \hat{n}_\alpha = n$. If $m \in Z_1$ then the map $n \longrightarrow m\Box n$ weak*-weak* continuous on $\mathfrak{A}^{**}$. So,

$$m\Box n = w^* - \lim_a lpham\Box\hat{n}_\alpha = w^* - \lim_a lpham\Diamond\hat{n}_\alpha = m\Diamond n.$$

Conversely. Suppose that for all $n \in \mathfrak{A}^{**}$, $m\Box n = m\Diamond n$. So,

$$m\Box n = m\Diamond n = w^* - \lim_\alpha m\Diamond\hat{n}_\alpha = w^* - \lim_\alpha m\Box\hat{n}_\alpha$$

Hence, the map $n \longrightarrow m\Box n$ is weak*-weak* continuous, so, $m \in Z_1$. $\Box$

Corollary 2. *Let $\mathfrak{A}$ be a Banach algebra and $n \in \mathfrak{A}^{**}$ such that $n\Box\hat{\mathfrak{A}} = \hat{\mathfrak{A}}\Box n$ then*

(i) *$n \in Z_1$ (resp. $n \in Z_2$) if and only if $n\Box m = m\Box n$ (resp. $n\Diamond m = m\Diamond n$) for all $m \in \mathfrak{A}^{**}$.*

(ii) *If $\mathfrak{A}$ is commutative then $Z_1 = Z_2$.*

Proof. Suppose that $m \in \mathfrak{A}^{**}$ and (m_α) is a net in $\mathfrak{A}$ so that converges weak* to m. So, we have

$$\begin{aligned} m\Box n &= w^* - \lim_\alpha \hat{m}_\alpha\Box n = w^* - \lim_\alpha n\Box\hat{m}_\alpha \\ &= w^* - \lim_\alpha n\Diamond\hat{m}_\alpha = n\Diamond m. \end{aligned}$$

By lemma 1, the result follows.

(ii) if $\mathfrak{A}$ is commutative then $n\Box m = m\Diamond n$. $\Box$

For a Banach algebra $\mathfrak{A}$, the $\mathfrak{A}^{op}$ is the algebra obtained by reversing the order of multiplication in $\mathfrak{A}$; i.e. for a, b in $\mathfrak{A}, \mathfrak{A}^{op}$ has the product "$\circ$" by

$$a \circ b = ba$$

For $m, n \in \mathfrak{A}^{**}$, the first and second Arens multiplication in $(\mathfrak{A}^{**})^{op}$, are indicated by $m\Box^{op}n = n\Box m$, $m\Diamond^{op}n = n\Diamond m$.

Theorem 3. *Let $\mathfrak{A}$ be a Banach algebra. Then, $\mathfrak{A}$ is Arens regular, amenable, weakly amenable if and only if is so $\mathfrak{A}^{op}$*

Proof. It is clear that $\mathfrak{A} = (\mathfrak{A}^{op})^{op}$. So, the proof of the converse implication is clear.

Let $\mathfrak{A}$ be Arens regular and n, m in $\mathfrak{A}^{**}$. Suppose (n_α) (m_β) be two nets in$\mathfrak{A}$ which converge to n, m in the weak* topology of $\mathfrak{A}^{**}$, respectively. We have

$$\begin{aligned} n\Box m &= w^* - \lim_\alpha w^* - \lim_\beta \widehat{(n_\alpha m_\beta)} \\ &= w^* - \lim_\alpha w^* - \lim_\beta \widehat{(m_\beta \circ n_\alpha)} = m\Diamond n = n\Diamond^{op} m \end{aligned}$$

Similarly, $n\Diamond m = n\Box^{op} m$. So, we have

$$(\mathfrak{A}^{**}, \Box) = (((\mathfrak{A}^{op})^{**})^{op}, \Diamond), \quad (\mathfrak{A}^{**}, \Diamond) = (((\mathfrak{A}^{op})^{**})^{op}, \Box)$$

Hence, $\mathfrak{A}^{op}$ is Arens regular.

Now, let $\mathfrak{A}$ be amenable and X^{op} be a Banach $\mathfrak{A}^{op}$-bimodule. We define a constracture on X which shows that X is a Banach $\mathfrak{A}$-bimodule. We need some definitions:

For $a, b \in \mathfrak{A}$ and x in X

$$< a.x, b >=< x, a \circ b >, < x.a, b >=< x, b \circ a >$$

so, we have, $a.x = x \circ a$ and $x.a = a \circ x$ and then

$$a.(b.x) = a.(x \circ b) = (x \circ b) \circ a = x \circ (b \circ a) = x \circ (ab) = (ab).x$$

Similarly, $(x.b).a = x.(ba)$, $(a.x).b = a.(x.b)$. So, X is a Banach $\mathfrak{A}$-bimodule. Suppose that $D : \mathfrak{A}^{op} \longrightarrow (X^{op})^*$ is a bounded derivation. We define $\Delta : \mathfrak{A} \longrightarrow X$ by

$$\Delta(ab) = D(b \circ a).$$

Hence, Δ is a bounded derivation; because

$$\begin{aligned} \Delta(ab) &= D(b \circ a) = D(b) \circ a + b \circ D(a) \\ &= D(a).b + a.D(b) = \Delta(a).b + a.\Delta(b) \end{aligned}$$

$\mathfrak{A}$ is amenable. So, there is f in X^* such that

$$\Delta(a) = a.f - f.a$$

so, we have

$$D(a) = \Delta(a) = f \circ a - a \circ f = a \circ (-f) - (-f) \circ a$$

so, D is inner. Therefor, $\mathfrak{A}^{op}$ is amenable.

Suppose that $\mathfrak{A}$ is weakly amenable. We know that $\mathfrak{A}^*$ is a Banach $\mathfrak{A}$-bimodule in natural way, we shall also consider that $(\mathfrak{A}^{**})^*$ is also $\mathfrak{A}^{op}$-bimodule.

For a, b in $\mathfrak{A}$, $f \in \mathfrak{A}^*$, we know that af and fa belong to $\mathfrak{A}^*$. Now, we define

$$< a \circ f, b >=< f, ab >, < f \circ a, b >=< f, ba > .$$

So, $fa = a \circ f$, $af = f \circ a$. If $D : \mathfrak{A}^{op} \longrightarrow (\mathfrak{A}^{op})^*$ is a bounded derivation then $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A}^*$ define by

$$\Delta(ab) = D(b \circ a)$$

is a bounded derivation. $\mathfrak{A}$ is weakly amenable. So, Δ is inner. There is a $f \in \mathfrak{A}^*$ such that,

$$D(a) = af - fa \quad \text{for } a \in \mathfrak{A}.$$

So, we have

$$\Delta(a) = a \circ (-f) - (-f) \circ a.$$

Hence, $\mathfrak{A}^{op}$ is weakly amenable. $\square$

It can be wonder, if amenability is also linked to Arens regularity. If we have amenability $\mathfrak{A}^{**}$ implies Arens regularity $\mathfrak{A}$, then there is nothing to prove. So, we have the following theorem. The proof of theorem followed the method of [3], Theorem 1.3.

Theorem 4. *Let $\mathfrak{A}$ be a Banach algebra such that $\mathfrak{A}^{**} = Z_1 \oplus I$ (resp. $\mathfrak{A}^{**} = Z_2 \oplus I$) for a weak*-closed ideal I. If $(\mathfrak{A}^{**}, \Box)$ (resp. $(\mathfrak{A}^{**}, \Diamond)$) is amenable then $\mathfrak{A}$ is Arens regular.*

Proof. We shall only give the proof of assertion $(\mathfrak{A}^{**}, \Box)$, that of $(\mathfrak{A}^{**}, \Diamond)$ is very similar. Since I is a complemented ideal in the amenable algebra $\mathfrak{A}^{**}$, it is itself amenable. So, I has a bounded approximated identity (e_i). Let E be a weak*-cluster point of (e_i) in $\mathfrak{A}^{**}$. Without loss of generality, we can assume that (e_i) converges weak* to E. By hypothesis $E \in I$. If $n \in I$,

$$E \Box n = w^* - \lim e_i \Box n = \lim e_i \Box n = n$$

$$n \Box E = \lim_i (n \Box E) \Box e_i = \lim_i n \Box (E \Box e_i) = \lim_i n \Box e_i = n$$

So, E is an identity for I. For all $n \in \mathfrak{A}^{**}$, $E \Box n$ and $n \Box E$ belong to I. Thus, for $n \in \mathfrak{A}^{**}$

$$E \Box n = (E \Box n) \Box E = E \Box (n \Box E) = n \Box E$$

By Goldstme's Theorem, we can find (n_α) in $\mathfrak{A}$ converges weak* to n. Thus

$$E \Box n = n \Box E = w^* - \lim_\alpha \hat{n}_\alpha \Box E = w^* - \lim_\alpha E \Box \hat{n}_\alpha = w^* - \lim E \Diamond \hat{n}_\alpha = E \Diamond n$$

By the corollary 2(i), $E \in Z_1$. Thus, $E \in Z_1 \cap I = \{0\}$. Hence $E = 0$ and $I = \{0\}$. So, we have $\mathfrak{A}^{**} = Z_1$. $\Box$

Theorem 5. *Let $\mathfrak{A}$ be a Banach algebra and Z_1 (resp. Z_2) be a left or right ideal $\mathfrak{A}^{**}$. If $(\mathfrak{A}^{**}, \Box)$ (resp. $(\mathfrak{A}^{**}, \Diamond)$) is amenable then $\mathfrak{A}$ is Arens regular.*

Proof. Suppose that $\mathfrak{A}^{**}$ is amenable then $\mathfrak{A}^{**}$ has a bounded approximate identity. By ([3], Lemma 1.1) $\mathfrak{A}^{**}$ has an identity, say E. Let $n \in \mathfrak{A}^{**}$. By using Goldestine's Theorem, we can find a net (n_α) in $\mathfrak{A}$ with converges weak* to n. So, by weak*-continuous second Arens multiplication, we have

$$
\begin{aligned}
E\Diamond n &= w^* - \lim E\Diamond \hat{n}_\alpha = w^* - \lim_\alpha E\Box \hat{n}_\alpha \\
&= w^* - \lim_\alpha \hat{n}_\alpha = n
\end{aligned}
$$

Therefore, for all $n \in \mathfrak{A}^{**}$, $E\Box n = E\Diamond n$. By lemma 1, $E \in Z_1$. By hypothesis, Z_1 is a left or right ideal of $\mathfrak{A}^{**}$, then $Z_1 = \mathfrak{A}^{**}$. Hence, $\mathfrak{A}$ is Arens regular. $\Box$

If $\mathfrak{A}$ is a commutative Banach algebra then the following theorem has given the links made between the amenability or weakly amenability $(\mathfrak{A}^{**}, \Box)$ and the amenability or weakly amenability $(\mathfrak{A}^{**}, \Diamond)$.

Theorem 6. *Let $\mathfrak{A}$ be a commutative Banach algebra. Then* $(\mathfrak{A}^{**}, \Box)$ *is amenable (weakly amenable) if and only if* $(\mathfrak{A}^{**}, \Diamond)$ *is amenable (weakly amenable).*

Proof. For $n \in \mathfrak{A}^{**}$, $m \in \mathfrak{A}^{**}$, we take two nets (n_α) and (m_β) in $\mathfrak{A}$ which converge to n, m in the weak*-topology in $\mathfrak{A}^{**}$, respectively. Then

$$
\begin{aligned}
n\Box m &= w^* - \lim_\alpha w^* - \lim_\beta (n_\alpha m_\beta)\hat{} \\
&= w^* - \lim_\alpha w^* - \lim_\beta (m_\beta n_\alpha)\hat{} \\
&= m\Diamond n
\end{aligned}
$$

Similarly, $n\Diamond m = m\Box n$. So, we have,

$$
(\mathfrak{A}^{**}, \Box) = ((\mathfrak{A}^{**})^{op}, \Diamond), (\mathfrak{A}^{**}, \Diamond) = ((\mathfrak{A}^{**})^{op}, \Box)
$$

From Theorem 3 the implication of this theorem also follows. $\Box$

Now, let $\mathfrak{A}$ have a continuous involution. So, there is a continuous anti-homomorphism from $\mathfrak{A}$ into $\mathfrak{A}$.

The following lemma plays a key role in next Theorem.

Lemma 7. *Let $T : X \longrightarrow Y$ be a continuous surjective anti-homomorphism between two Banach spaces X and Y. Then*

$$T^{**} : (X^{**}, \Box) \longrightarrow ((Y^{**})^{op}, \Diamond), T^{**} : (X^{**}, \Diamond) \longrightarrow ((Y^{**})^{op}, \Box)$$

are also both continuous surjective, homomorphism

Proof. For $m, n \in (X^{**}, \Box)$, by using the Goldstine's theorem, we can find two bounded nets $(m_\alpha), (n_\beta)$ in X which converge to m, n in the weak* topology X^{**}. So, we have

$$\begin{aligned} T^{**}(m\Box n) &= w^* - \lim_\alpha w^* - \lim_\beta T^{**}(\hat{m}_\alpha \Box \hat{n}_\beta) \\ &= w^* - \lim_\alpha w^* - \lim_\beta (T(m_\alpha n_\beta))\hat{} \\ &= w^* - \lim_\alpha w^* - \lim_\beta (T(n_\beta))\hat{}\,\Box (T(m_\alpha))\hat{} \\ &= w^* - \lim_\alpha w^* - \lim_\beta T^{**}(\hat{n}_\beta) \Diamond T^{**}(\hat{m}_\alpha) \\ &= T^{**}(n) \Diamond T^{**}(m) = T^{**}(m) \Diamond^{op} T^{**}(n) \end{aligned}$$

Similarly, $T^{**}(m \Diamond n) = T^{**}(m) \Box^{op} T^{**}(n)$.

Now, we shows that T^{**} is surjective. If $m \in Y^{**}$, the Goldstine's theorem gives a bounded net (m_α) in Y which converges to m in the weak* topology. By the open mapping theorem, there is a bounded net (n_α) in X with $T(n_\alpha) = m_\alpha$. Suppose that n is a weak*-cluster point of $(\hat{n}_\alpha)$ in X^{**}. So, by weak*-weak* continuity of T^{**}, we have

$$T^{**}(n) = w^* - \lim_\alpha T^{**}(\hat{n}_\alpha) = w^* - \lim_\alpha (T(n_\alpha))\hat{} = w^* - \lim_\alpha \hat{m}_\alpha = m. \Box$$

Lemma 8. *Let $(X, \Box)$ and $(Y, \Diamond)$ be Banach algebras and let T be an anti-homomorphism of X on to a dense subset of Y. If $D : Y \longrightarrow Y^*$ is non-zero derivation then $\bar{D} = T^* D T$: $X \longrightarrow X^*$ is a noon-zero derivation:*

Proof. First we show that $\bar{D}$ is non-zero derivation. If $\bar{D} = 0$ then for all x, t in X,

$$< \bar{D}(x), t >= 0.$$

Thus,

$$0 =< T^*DT(x), t >=< DT(x), T(t) > .$$

So, by density $T(X)$ in Y, $DT(X) = 0$, and then, $D = 0$.

Now, we prove that $\bar{D}$ is a derivation. For a, b in X,

$$T(a\Box b) = T(b)\Diamond T(a)$$

suppose $x \in X$. We have

$$\begin{aligned}
< \bar{D}(a\Box b), x > &=< T^*DT(a\Box b), x > \\
&=< D(T(b)\Diamond T(a)), T(x) > \\
&=< DT(b)\Diamond T(a) + T(b)\Diamond DT(a), T(x) > \\
&=< DT(b)\Diamond T(a), T(x) > + < T(b)\Diamond DT(a), T(x) > \\
&=< DT(b), T(a)\Diamond T(x) > + < DT(a), T(x)\Diamond T(b) > \\
&=< DT(b), T(x\Box a) > + < DT(a), T(b\Box x) > \\
&=< T^*DT(b), x\Box a > + < T^*DT(a), b\Box x > \\
&=< a\Box\bar{D}(b), x > + < \bar{D}(a)\Box b, x > \\
&=< a\Box\bar{D}(b) + \bar{D}(a)\Box b, x > .
\end{aligned}$$

Thus,

$$\bar{D}(a\Box b) = \bar{D}(a)\Box b + a\Box\bar{D}(b)$$

we conclude that $\bar{D}$ is a derivation $\Box$

Lemma 9. *Let $\mathfrak{A}$ be a dense subset in the Banach algebra of $\mathcal{B}$. Then $\mathfrak{A}^{**}$ is w^*-dense in $\mathfrak{B}^{**}$.*

Proof. Suppose that $n \in \mathfrak{B}^{**}$. Then $n = w^* - \lim_\alpha \hat{n}_\alpha$, for some a net (n_α) in $\mathfrak{B}$. Since, $\mathfrak{A}$ is dense in $\mathfrak{B}$, there is a net $(m_{(\alpha,\beta)})_\beta$ in $\mathfrak{A}$ such that

$$\lim_\beta m_{(\alpha,\beta)} = n_\alpha$$

Now, for each open neighborhood U of n_α, there is $\alpha(U)$ such that for each $\beta > \alpha(U)$, we have

$$m_{(\alpha,\beta)} \in U$$

Turn the family $(m_{(\alpha,\beta,u)})$ into a net by directing the index set in the obvious way. Then $m_{(\alpha,\beta,u)} \in \mathfrak{A}$ and

$$w^* - \lim_{(\alpha,\beta,u)} \hat{\alpha}_{(\alpha,\beta,u)} = n$$

Hence, $\mathfrak{A}^{**}$ is w^*-dense in $\mathfrak{B}^{**}$. $\square$

Theorem 10. *Let $\mathfrak{A}$ be a Banach algebra and $T : \mathfrak{A} \longrightarrow \mathfrak{A}$ be continuous an anti-homomorphism.*

(i) *If the range T is dense in $\mathfrak{A}$ then $(\mathfrak{A}^{**}, \square)$ is amenable if and only if $(\mathfrak{A}^{**}, \square)$ is amenable.*

(ii) *If $TT(x) = x$ then $(\mathfrak{A}^{**}, \square)$ is weakly amenable if and only if $(\mathfrak{A}^{**}, \diamond)$ is weakly amenable.*

Proof. (i). Suppose that T^{**} is second adjoint of T. So, T^{**} is weak*-weak* continuous and by Lemma 7,

$$T^{**} : (\mathfrak{A}^{**}, \square) \longrightarrow ((\mathfrak{A}^{**})^{op}, \diamond)$$
$$T^{**} : (\mathfrak{A}^{**}, \diamond) \longrightarrow ((\mathfrak{A}^{**})^{op}, \square)$$

and T^{**} is a continuous homomorphism. By the Lemma 9, the rang of T^{**} is weak* dense in $\mathfrak{A}^{**}$. We conclude from ([6], Theorem 5.3), $(\mathfrak{A}^{**}, \square)$ [resp. $(\mathfrak{A}^{**}, \diamond)$] is amenable if and only if $((\mathfrak{A}^{**})^{op}, \diamond)$ [resp. $((\mathfrak{A}^{**})^{op}, \square)$ is so. By Theorem 3, $(\mathfrak{A}^{**}, \diamond)$ [resp. $(\mathfrak{A}^{**}, \square)$] is amenable.

Suppose that $(\mathfrak{A}^{**}, \square)$ is weakly amenable. By Theorem 3, $((\mathfrak{A}^{**})^{op}, \square)$ is weakly amenable. It is clear that

$$(((\mathfrak{A}^{**})^{op})^{*}, \square) = ((\mathfrak{A}^{***})^{op}, \square)$$
$$(((\mathfrak{A}^{**})^{op})^{*}, \diamond) = ((\mathfrak{A}^{***})^{op}, \diamond)$$

Because, if $\xi \in \mathfrak{A}^{***}$ and $m \in \mathfrak{A}^{**}$ then; $m \square^{op} \xi \in ((\mathfrak{A}^{***})^{op}, \square)$ if and only if $\xi \square m \in (\mathfrak{A}^{***}, \square)$ if and only if $m \square^{op} \xi \in (((\mathfrak{A}^{**})^{op})^{*}, \square)$.

Now, let D be a derivation from $(\mathfrak{A}^{**}, \diamond)$ into $(\mathfrak{A}^{***}, \diamond)$. Then

$$\begin{tikzcd}
(\mathfrak{A}^{**}, \diamond) \arrow[r, "D"] \arrow[d, "T^{**}"] & (\mathfrak{A}^{***}, \diamond) \arrow[d, "T^{***}"] \\
((\mathfrak{A}^{**})^{op}, \square) \arrow[r, "\overline{D}"] & (((\mathfrak{A}^{**})^{op})^{*}, \square) = ((\mathfrak{A}^{***})^{op}, \square)
\end{tikzcd}$$

By lemma 8, we conclude that $\bar{D} = T^{***} D T^{**}$ is a derivation from $((\mathfrak{A}^{**})^{op}, \square)$ into $(((\mathfrak{A})^{**})^{op})^{*}, \square)$. By $((\mathfrak{A}^{**})^{op}, \square)$ is amenable. So, there is $\xi \in ((\mathfrak{A}^{**})^{op})^{*} = (\mathfrak{A}^{***})^{op}$ such that for all $n \in ((\mathfrak{A}^{**})^{op}, \square)$

$$\bar{D}(n) = n \square^{op} \xi - \xi \square n^{op} = \xi \square n - n \square \xi$$

Since, $TT = I$. We have $T^{***}T^{***} = I^{***}$, $T^{**}T^{**} = I^{**}$. If $n \in (\mathfrak{A}^{**}, \diamond)$ then $T^{**}(n) \in ((\mathfrak{A}^{**})^{op}, \square)$, $T^{**}T^{**}(n) \in (\mathfrak{A}^{**}, \diamond)$. Also,

$$D(n) \in (\mathfrak{A}^{***}, \diamond), T^{***}(D(n)) \in ((\mathfrak{A}^{***})^{op}, \square),$$
$$T^{***}T^{***}(D(n)) \in (\mathfrak{A}^{***}, \diamond).$$

we claim that,

$$(I) \quad T^{***}(\xi \square T^{**}(n)) = n \diamond T^{***}(\xi)$$
$$(II) \quad T^{***}(T^{**}(n) \square \xi) = T^{***}(\xi) \diamond n$$

we prove (I), it is routine to check that (II). For $m \in (\mathfrak{A}^{**}, \Diamond)$

$$\begin{aligned} < T^{***}(\xi \Box T^{**}(n)), m > &= < \xi \Box T^{**}(n), T^{**}(m) > \\ &= < \xi, T^{**}(n) \Box T^{**}(m) > \\ &= < \xi, T^{**}(m \Diamond n) > \\ &= < T^{***}(\xi), m \Diamond n > \\ &= < n \Diamond T^{***}(\xi), m > \end{aligned}$$

Hence, we have (1). Now, suppose that $n \in (\mathfrak{A}^{**}, \Diamond)$ So,

$$\begin{aligned} D(n) &= T^{***}((T^{***} D T^{**}) T^{**}(n)) \\ &= T^{***}(\bar{D} T^{**}(n)) \\ &= T^{***}(\xi \Box T^{**}(n) - T^{**}(n) \Box \xi) \\ &= T^{***}(\xi \Box T^{**}(n)) - T^{***}(T^{**}(n) \Box \xi) \\ &= n \Diamond T^{***}(\xi) - T^{***}(\xi) \Diamond n \end{aligned}$$

So, D is inner. Thus $(\mathfrak{A}^{**}, \Diamond)$ weakly amenable The converse is similar. $\Box$

Corollary 11. *Let* $\mathfrak{A}$ *be a Banach algebra with continuous involution then* $(\mathfrak{A}^{**}, \Box)$ *is amenable (weakly amenable) if and only if* $(\mathfrak{A}^{**}, \Diamond)$ *is amenable (weakly amenable).*

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SUBLINEAR FUNCTION ON TOPOLOGICAL GROUPS

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ABSTRACT. In this article, sublinear functions of R^n are generalized to topological groups and their relations are inspected with convex functions. These functions are generalization of homomorphism too, along the above notable objects are mentioned.

1

1. Introduction

A linear function T from R^n to R or a linear form on R^n is primarily defined as a function satisfying for all $(x_1, x_2) \in R^n \times R^n$ and $(t_1, t_2) \in R \times R$

$$T(t_1x_1 + t_2x_2) = t_1T(x_1) + t_2T(x_2) \qquad (1)$$

A corresponding definition for a sublinear function f from R^n into R is: for all $(x_1, x_2) \in R^n \times R^n$ and $(t_1, t_2) \in R^+ \times R^+$

$$f(t_1x_1 + t_2x_2) \le t_1f(x_1) + t_2f(x_2) \qquad (2)$$

Definition 1. A function $f : R^n \to (-\infty, +\infty]$ is said to be sublinear if it is convex and positively homogeneous (of degree 1): i.e. $f \in \mathrm{conv}(R^n)$

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and

$$f(tx) = tf(x) \text{ for all } x \in R^n \text{ and } t > 0. \tag{3}$$

Remark 2. Inequality in (3) would be enough to define positive homogeneity: a function f is positivety homogeneous if and only if it satisfies

$$f(tx) \le tf(x) \text{ for all } x \in R^n \; and \; t > 0 \tag{4}$$

In fact (4) implies ($tx \in R^n$ and $t^{-1} > 0$)

$$f(x) = f(t^{-1}tx) \le t^{-1}f(tx)$$

which together with (4) shows that f is positively homogeneous. The following result is a geometrical characterization of sublinear functions.
Proposition 3. *A function $f : R^n \to (-\infty, \infty]$ is sublinear if and only if its epigraph $epi(f)$ is a nonempety convex cone in $R^n \times R$.*
Proposition 4. *A function $f : R^n \to (-\infty, \infty]$ not identically equal to ∞, is sublinear if and only if one of the following two properties holds:*

$$f(t_1x_1 + t_2x_2) \le t_1f(x_1) + t_2f(x_2) \text{ for all } x_1, x_2 \in R^n \text{ and } t_1, t_2 > 0 \tag{5}$$

or f is positively homogeneous and subadditive.
Corollary 5. *If f is sublinear then*

$$f(x) + f(-x) \ge 0 \text{ for all } x \in R^n. \tag{6}$$

Proposition 6. *Let f be sublinear and suppose that there exist $x_1, x_2, ..., x_m$ in D_f such that*

$$f(x_j) + f(-x_j) = 0 \text{ for } j = 1, 2, ..., m \tag{7}$$

then f is linear on the subspace spanned by $x_1, x_2, ..., x_m$.
Definition 7. Let S be a nonempty set in R^n the function $f_s : R^n \to (-\infty, \infty]$ defined by

$$f_s(x) = sup\{< s, x >; s \in S\}$$

is called the support functions of S.

Proposition 8. *A support function is sublinear.*

2. Sublinear function on groups

In this section, some properties of subadditivity and sublinearity functions will be inspected which are shown in the introduction on topological groups.[1]

Definition 1. Let G be a group and Ω is an open set. A function $f : \Omega \to (-\infty, \infty]$ is subadditive if for all $x_1, x_2 \in \Omega$ such that $x_1x_2 \in \Omega$,

$$f(x_1x_2) \leq f(x_1) + f(x_2) \tag{8}$$

and f is midhomogeneity, if for all $a \in \Omega$ such that a^2 in Ω,

$$f(a^2) = 2f(a) \tag{9}$$

the f is midsublinear if is subadditive and midhomogeneity.

Remark. Rgarding to (9) by induction we show that $f(x^{2^n}) = 2^n f(x)$ when $x, x^2, x^4, ..., x^{2^n} \in \Omega$. Let m be an integer then an integer n exist such that $m < 2^n$ thus,

$$2^n f(x) = f(x^{2^n}) = f(x^m x^{2^n - m})$$

$$\leq f(x^m) + f(x^{2^n - m}) \leq f(x^m) + (2^n - m)f(x)$$

$$mf(x) \leq f(x^m)$$

on the other hand $f(x^m) \leq mf(x)$ therefore $mf(x) = f(x^m)$. In addition, if G has square root property [1], [2] (i.e. for all $a \in G$ there exist $b \in G$ such that $b^2 = a$. For example $Gl(n; C)$ has square root property.) hence for all $x \in G$ and $n \in N$ there exist a $y \in G$ such that $y^{2^n} = x$ thus we define $y = x^{1/2^n}$,since $f(x) = f(y^{2^n}) = 2^n f(y)$ therefore

$$f(x^{1/2^n}) = \frac{1}{2^n} f(x), \text{ and } f(x^{m/2^n}) = \frac{m}{2^n} f(x).$$

If G is rootapproximable then for all $t > 0$ there exists a sequence $\{q_i\}$ of the form $q_i = \frac{m_i}{2^{n_i}}$ such that $q_i \to t$.We define $x^t := lim_{i\to\infty} x^{q_i}$. If f is midsublinear and continuous function then

$$f(x^t) = lim_{i\to\infty} f(x^{q_i}) = lim_{i\to\infty} q_i f(x) = tf(x).$$

Definition 2. A function $f : \Omega \to (-\infty, \infty]$ is sublinear if is midsublinear and continuous on Ω.

Proposition 3. *Let G be a rootapproximable,abelian group, and Ω is an open set in G, $f : \Omega \to (-\infty, \infty]$ midconvex and midhomogeneity then f is midsublinear.*

Proof. It's sufficient to show that f is subadditive,

$$f(x+y) = 2f(\frac{x+y}{2}) \le 2[f(\frac{x}{2})+f(\frac{y}{2})] = 2(\frac{f(x)+f(y)}{2}) = f(x)+f(y).$$

Since $f(e) = f(2e) = 2f(e)$ thus $f(e) = 0$ or $f(e) = \infty$ hence $0 \le f(e) = f(xx^{-1}) \le f(x) + f(x^{-1})$. If $f(e) = 0$, define $H = \{x; x \in G, f(x) + f(x^{-1}) = 0\}$ then $H \le G$ because, if $x, y \in H$ then $f(x) + f(x^{-1}) = f(y) + f(y^{-1}) = 0$

$$0 \le f(xy^{-1}) + f((xy^{-1})^{-1}) = f(xy^{-1}) + f(yx^{-1})$$

$$\le f(x) + f(y^{-1}) + f(y) + f(x^{-1}) = 0$$

therefore $f(xy^{-1}) + f((xy^{-1})^{-1}) = 0$ and then $xy^{-1} \in H$.

Hence the following results:

Proposition 4. *suppose $f : G \to (-\infty, \infty]$ is midsublinear function and $f(e) = 0$, then $H = \{x; f(x) + f(x^{-1}) = 0\}$ is sublinear of G.*

Corollary. *If $f : G \to R$ is midsublinear and H is above subgroup, then we have for all $x, y \in H$, $f(xy) = f(x) + f(y)$.*

Because $f(y) = f(yxx^{-1}) \le f(yx) + f(x^{-1}) = f(yx) - f(x)$ and hence f on H is linear.

Proposition 5. *If $f : G \to R$ be a midsublinear, such that $H = \{e\}$,* *define* $||x||_f := \text{Max}\{f(x), f(x^{-1})\}$ *then* $||.||_f$ *has the following properties:*

i) $||x||_f \geq 0$ *for all* x *in* G

ii) $||x||_f = 0$ *iff* $x = e$

iii) $||x^t||_f = |t|\ ||x||_f$ *(In the case* G *is rootapproximable and* f *sublinear.)*

iv) $||xy||_f \leq ||x||_f + ||y||_f$

Remark. If G is abelian group, then tx and $x + y$ substitute with x^t and xy respectively.

Proof.

$$||x^t||_f = \text{Max}\{f(x^t), f(x^{-t})\} =$$

$$\text{Max}\{tf(x), tf(x^{-1})\} \text{ if } t \geq 0$$

$$\text{Max}\{-tf(x^{-1}), -tf(x)\} \text{ if } t < 0$$

therefore

$$||xy||_f = \text{Max}\{f(xy), f(y^{-1}x^{-1})\} \leq \text{Max}\{f(x) + f(y), f(x^{-1}) + f(y^{-1})\}$$

$$\leq \text{Max}\{f(x), f(x^{-1})\} + \text{Max}\{f(y), f(y^{-1})\} = ||x||_f + ||y||_f.$$

Proposition 6. *Let* $\{\varphi_\alpha\}$ *be the collection of midsublinear (res. sublinear) function on* G *then* $\varphi = \text{sup}\varphi_\alpha$ *is midsublinear (res. sublinear) function on* G*.*

Proof. Since for all α, $\varphi_\alpha(xy) \leq \varphi_\alpha(x) + \varphi_\alpha(y) \leq \varphi(x) + \varphi(y)$ thus $\varphi(xy) \leq \varphi(x) + \varphi(y)$ and $\varphi_\alpha(x^t) = t\varphi_\alpha(x)$ hence $\varphi(x^t) = t\varphi(x)$.

Example. Suppose that $G = Gl(n; C)$ $\varphi(A) = log|det(A)|$ then $\varphi(AB) = \varphi(A) + \varphi(B)$ and $\varphi(A^q) = q\varphi(A)$ for all $q \in Q^+$ since φ is continuous therefore $\varphi(A^t) = t\varphi(A)$ for all $t > 0$ [2] hence φ is linear function.

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ON COMPACTNESS AND WEAKLY COMPACTNESS OF THE BEST APPROXIMANT SET

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Abstract. Weakly-Chebyshev subspaces of a Banach space X are defined as those in which the set of best approximants of any vector x in X is non-empty and weakly compact. Moreover, it is shown that there exists a proximinal subspace G of X which is weakly-Chebyshev, but is not quasi-Chebyshev in X. Also, some other related results are presented.

1. Introduction and Preliminaries

Let X be a (complex or real) Banach space and let G be a linear subspace of X. A point $y_0 \in G$ is said to be a best approximation for $x \in X$ if

$$\|x - y_0\| = d(x, G) = \inf\{\|x - y\| : y \in G\}.$$

If each $x \in X$ has at least one best approximation in G, then G is called a proximinal subspace of X. If each $x \in X$ has a unique best approximation in G, then G is called a Chebyshev subspace of X. For $x \in X$, put

$$P_G(x) = \{y \in G : \|x - y\| = d(x, G)\},$$

and

$$\hat{G} = P_G^{-1}(0) = \{x \in X : \|x\| = d(x, G)\}.$$

It is clear that $P_G(x)$ is a closed, bounded and convex subset of X. For an arbitrary non-empty convex set A in X we shall denote by

$$\ell(A) = \{\alpha x + (1-\alpha)y : x, y \in A; \alpha \text{ is scalar}\}$$

the linear manifold spanned by A. For every fixed $y \in A$ the set $\ell(A) - y = \{x - y : x \in \ell(A)\}$ is a linear subspace of X satisfying $\ell(A - y) = \ell(A) - y$. The dimension of A is defined by $dimA = dim\ell(A)$. Then, for every $y \in A$ we have

$$dimA = dim\ell(A) = dim[\ell(A) - y] = dim\ell(A - y) = dim(A - y).$$

(For more details see [10].)

We say that G is a pseudo-Chebyshev subspace of X if $P_G(x)$ is a non-empty and finite-dimensional set in X for every $x \in X$.

We say that G is a quasi-Chebyshev subspace of X if $P_G(x)$ is a non-empty and compact set in X for every $x \in X$. Every pseudo-Chebyshev subspace is quasi-Chebyshev, but the converse is not true (see [4]). The properties of pseudo-Chebyshev and quasi-Chebyshev subspaces have investigated in [4], [5], [6], [7] and [9].

In the following we give a definition which extends the definition of quasi-Chebyshev subspace.

1.1. Definition. Let X be a Banach space. A linear subspace G of X is called weakly-Chebyshev if $P_G(x)$ is non-empty and weakly compact set in X for every $x \in X$.

It is clear that every quasi-Chebyshev subspace is weakly-Chebyshev. In the following we shall give an example in which the converse is not true.

Let X^* be the dual space of the Banach space X. For $0 \neq f \in X^*$, put

$$M_f = \{x \in X : f(x) = \|f\|, \|x\| = 1\}.$$

We say that a Banach space X is quasi-strictly convex if for every $0 \neq f \in X^*$ the set M_f is compact (see [7]). Every k-strictly convex Banach space is quasi-strictly convex ($k = 1, 2, \dots$).

We recall that for a topological space Y one denotes by 2^Y the collection of all bounded closed subsets of Y. A mapping $\mathcal{U} : X \longrightarrow 2^Y$ is called upper semi-continuous (u.s.c.) if the set

$$\{x \in X : \mathcal{U}(x) \subset M\}$$

is open for every open subset M of Y, or the set

$$\{x \in X : \mathcal{U}(x) \cap N \neq \emptyset\}$$

is closed for every closed subset N of Y. (For more details see [10].)

It is clear that P_G is upper semi-continuous if and only if for every closed subset A of G the set $A + \hat{G}$ is closed, or if and only if for every closed subset A of $\hat{G}$ the set $G + A$ is closed (see [2]).

We conclude this section by a list of known lemmas needed in the proof of the main results.

1.2. Lemma ([9,10]). *Let X be a normed linear space, G be a linear subspace of $X, x \in X \backslash \overline{G}$ and F be a subset of G. Then F is a subset of $P_G(x)$ if and only if there exists $f \in X^*$ such that $\|f\| = 1$, $f|_G = 0$ and $f(x - y) = \|x - y\|$ for every $y \in F$.*

1.3. Lemma ([4]). *Let X be a Banach space and let G be a proximinal subspace of X with codimension one, then the following are equivalent:*

1) G is quasi-Chebyshev in X.

2) Each sequence $\{y_n\}_{n \geq 1}$ in X with $\|y_n\| = 1$ and $0 \in P_G(y_n)$ ($n = 1, 2, \dots$) has a convergent subsequence.

1.4. Lemma ([4]). *Let X be a Banach space and let G be a proximinal subspace of X. Then the following are equivalent:*

1) G is quasi-Chebyshev in X.

2) In every linear subspace $Y_x \subset X$ $(x \in X\backslash G)$ of the form $Y_x = G \oplus \langle x \rangle$ each sequence $\{y_n\}_{n\geq 1}$ in Y_x with $\|y_n\| = 1$ and $0 \in P_G(y_n)$ $(n = 1, 2, \dots)$ has a convergent subsequence.

1.5. Lemma ([7]). *Let X be a Banach space. Then all closed linear subspaces of X are quasi-Chebyshev if and only if X is reflexive and quasi-strictly convex.*

2. Characterizations of Weakly-Chebyshev Subspaces

In this section we shall give a characterization of weakly-Chebyshev subspaces in Banach spaces.

2.1. Theorem. *Let X be a Banach space, then all closed linear subspaces of X are weakly-Chebyshev if and only if X is reflexive.*

Proof. Suppose that all closed linear subspaces of X are weakly-Chebyshev, then all closed linear subspaces of X are proximinal, and hence by [5; Corollary 2.4] X is reflexive.

Conversely, assume that X is reflexive and G is any closed linear subspace of X. Then S_X the unit ball of X and hence $rS_X = \{x \in X : \|x\| \leq r\}$ is weakly compact for every $r > 0$.

Let $x \in X\backslash G$. Since $P_G(x)$ is a bounded set, therefore there exists $r > 0$ such that $P_G(x) \subseteq rS_X$. But $P_G(x)$ is weakly closed because it is closed and convex. Then $P_G(x)$ is weakly compact. It follows that G is a weakly-Chebyshev subspace of X.$\blacksquare$

We know that every quasi-Chebyshev subspace of X is weakly-Chebyshev subspace of X. The following example shows that there exists a weakly-Chebyshev subspace of a Banach space X which is not quasi-Chebyshev.

2.2. Example. Let $W = \ell^2$ with the basis $\{e_n\}_{n\geq 1}$ and let $W_0 = \ell^\infty$. Put, $X = W \oplus W_0$ and define a norm on X by

$$\|x + y\| = \max\{\|x\|_2, \|y\|_\infty\},$$

for all $x \in W$ and all $y \in W_0$.

It is clear that $\|\cdot\|$ is a norm on X, and X is a Banach space with respect to this norm. It is not difficult to show that

$$P_W(y) = \{x \in W : \|x\|_2 \leq \|y\|_\infty\},$$

for every $y \in W_0$. Let $z \in X$, $z = x + y$, where $x \in W$ and $y \in W_0$. Since $x \in W$, $P_W(z) = P_W(y) + x$. It follows that $P_W(z) \neq \emptyset$ for all $z \in X$, and hence W is a proximinal subspace of X.

Since W is a reflexive subspace of X, it follows that W is weakly-Chebyshev.

Now, we show that W is not quasi-Chebyshev subspace of X. Let $y = \{(-1)^n\}_{n\geq 1}$. Then, we have $y \in W_0$ and

$$P_W(y) = \{x \in W : \|x\|_2 \leq 1\} = S_W$$

where S_W is the unit ball of W, and hence $P_W(y)$ is not compact because W is reflexive. Therefore, W is not quasi-chebyshev in X.

The following Theorem shows that in a reflexive Banach space which is not quasi-strictly convex, there exists a closed linear subspace of X which is weakly-Chebyshev but is not quasi-Chebyshev.

2.3. Theorem. *Suppose X is a reflexive Banach space, but is not quasi-strictly convex. Then there exists a subspace of X which is weakly-Chebyshev, but is not quasi-Chebyshev.*

Proof. Since X is not quasi-strictly convex, there exists $0 \neq f_0 \in X^*$ such that M_{f_0} is not compact. It follows that there exists a sequence

$\{y_n\}_{n\geq 1}$ in X without a convergent subsequence such that

$$f_0(y_n) = \|f_0\| \quad \text{and} \quad \|y_n\| = 1 \ \ , \ \ (n = 1, 2, \dots).$$

Put, $G_0 = ker f_0$. Then G_0 is a closed linear subspace of X. Let $x_0 = y_1$, $g_n = x_0 - y_{n+1}$ $(n = 1, 2, \dots)$ and $f = \dfrac{f_0}{\|f_0\|}$, then $\{g_n\}_{n\geq 1}$ is a sequence in G_0, $f \in X^*$, $\|f\| = 1$, $f|_{G_0} = 0$ and

$$f(x_0 - g_n) = f(x_0) = \|x_0\| = \|x_0 - g_n\|, \ (n = 1, 2, \dots).$$

By Lemma 1.2, $\{g_n\}_{n\geq 1}$ is a sequence in $P_{G_0}(x_0)$ without a convergent subsequence. Therefore, G_0 is not quasi-Chebyshev. But, X is reflexive and hence by Theorem 2.1, G_0 is weakly-Chebyshev.■

2.4. Theorem. *Let X be a Banach space and let G be a proximinal subspace of X. If M_f is weakly compact for every $0 \neq f \in G^{\perp}$, then G is weakly-Chebyshev in X.*

Proof. Let $x \in X \backslash G$ and $\{g_n\}_{n\geq 1}$ be an arbitrary sequence in $P_G(x)$. Then, by Lemma 1.2, there exists $f_0 \in X^*$, $\|f_0\| = 1$, $f_0|_G = 0$ and $f_0(x - g_n) = \|x - g_n\|$ $(n = 1, 2, \dots)$.

Let $x_n = x - g_n$ $(n = 1, 2, \dots)$, then $f_0(x_n) = \|x_n\| = f_0(x)$ for all $n \geq 1$. Put

$$z_n = \frac{x_n}{\|x_n\|} = \frac{x_n}{f_0(x)}, \ (n = 1, 2, \dots).$$

Then $\{z_n\}_{n\geq 1}$ is a sequence in M_{f_0}, $\|z_n\| = 1$ and $f_0(z_n) = 1 = \|f_0\|$.

Since M_{f_0} is weakly compact, hence there exists a weakly-convergent subsequence $\{z_{n_k}\}_{k\geq 1}$ of $\{z_n\}_{n\geq 1}$ such that $z_{n_k} \xrightarrow{w} z_0 \in M_{f_0}$. Therefore, we have $x_{n_k} \xrightarrow{w} z_0 f_0(x)$ and hence $g_{n_k} \xrightarrow{w} x - z_0 f_0(x) \in G$.

Now, we show that $g_0 = x - z_0 f_0(x) \in P_G(x)$. To do this, we have $f_0(x - g_0) = f_0(z_0 f_0(x)) = f_0(x)$ and we also have

$$\|x - g_0\| = \|z_0 f_0(x)\| = |f_0(x)| \|z_0\| = f_0(x).$$

Therefore, $f_0(x-g_0) = \|x-g_0\|$, $\|f_0\| = 1$ and $f_0|_G = 0$. Hence, by Lemma 1.2, $g_0 \in P_G(x)$. It follows that $\{g_n\}_{n\geq 1}$ have a weakly-convergent subsquence in $P_G(x)$. Thus, G is weakly-Chebyshev in X.■

2.5. Theorem. *Let X be a Banach space and let G be a proximinal subspace of X with codimension one. Then the following are equivalent:*

1) G is weakly-Chebyshev in X.

2) Each sequence $\{y_n\}_{n\geq 1}$ in X with $\|y_n\| = 1$ and $0 \in P_G(y_n)$ $(n = 1, 2, \dots)$ has a weakly-convergent subsequence.

3) M_f is weakly compact for every $0 \neq f \in G^{\perp}$.

Proof. 1)$\Rightarrow$ 2). Assume that G is weakly-Chebyshev in X, $\{y_n\}_{n\geq 1}$ is any sequence in X with $\|y_n\| = 1$ and $0 \in P_G(y_n)$. Since $codimG = 1$, there exists $x_0 \in X$ such that $X = G \oplus \langle x_0\rangle$. Therefore, there exist a sequence $\{z_n\}_{n\geq 1}$ in G and a sequence $\{\beta_n\}_{n\geq 1}$ of scalars (note that $\beta_n \neq 0$ for all $n = 1, 2, \dots$) such that

$$y_n = z_n + \beta_n x_0, \qquad (n = 1, 2, \dots)\ \cdot$$

Now, we have

$$\begin{aligned} d(x_0, G) &= d(\frac{1}{\beta_n} y_n - \frac{1}{\beta_n} z_n, G) \\ &= d(\frac{1}{\beta_n} y_n, G) = \frac{1}{|\beta_n|} d(y_n, G) \\ &= \frac{1}{|\beta_n|}\|y_n\| = \frac{1}{|\beta_n|},\ (*) \end{aligned}$$

and

$$\|x_0 + \frac{1}{\beta_n} z_n\| = \frac{1}{|\beta_n|}\|y_n\| = \frac{1}{|\beta_n|},$$

for all $n \geq 1$. It follows that $\{-\frac{1}{\beta_n} z_n\}_{n\geq 1}$ is a sequence in $P_G(x_0)$. Since $P_G(x_0)$ is weakly compact, $\{\frac{1}{\beta_n} z_n\}_{n\geq 1}$ has a weakly-convergent subsequence. Also, it follows from $(*)$ that $\{\beta_n\}_{n\geq 1}$ is a bounded sequence of scalars. Hence $\{z_n\}_{n\geq 1}$ has a weakly-convergent subsequence. Thus, $\{y_n\}_{n\geq 1}$ has a weakly-convergent subsequence in X.

2)$\Rightarrow$ 3). Suppose $0 \neq f \in G^{\perp}$ and $\{y_n\}_{n\geq 1}$ is an arbitrary sequence in M_f. Then we have

$$f(y_n) = \|f\| \quad \text{and} \quad \|y_n\| = 1, \ (n = 1, 2, \dots).$$

Let $f_0 = \dfrac{f}{\|f\|}$. It follows that $f_0 \in X^*$, $\|f_0\| = 1$, $f_0|_G = 0$ and $f_0(y_n) = 1 = \|y_n\|$ for all $n = 1, 2, \dots$. Then, by Lemma 1.2, $0 \in P_G(y_n)$ and $\|y_n\| = 1$ $(n = 1, 2, \dots)$. Now, by hypothesis, $\{y_n\}_{n\geq 1}$ has a weakly-convergent subsequence in X. That is, there exists $\{y_{n_k}\}_{k\geq 1}$ such that $y_{n_k} \xrightarrow{w} y_0 \in X$. Since M_f is closed and convex, M_f is weakly closed and hence $y_0 \in M_f$. Then, M_f is weakly compact.

3)$\Rightarrow$ 1). This is a consequence of Theorem 2.5.■

2.6. Theorem. *Let X be a Banach space and let G be a proximinal subspace of X. Then the following are equivalent:*

1) G is weakly-Chebyshev in X.

2) For every $x \in X \backslash G$ and every $f \in X^*$ there exists $y_0 \in P_G(x)$ such that $|f(y)| \leq |f(y_0)|$ for all $y \in P_G(x)$.

3) There do not exist $f \in X^*$, $x_0 \in X$ and a sequence $\{x_n\}_{n\geq 1}$ in X without a weakly convergent subsequence and with $x_0 - x_n \in G$ $(n = 1, 2, \dots)$ such that $\|f\| = 1$, $f|_G = 0$ and $f(x_n) = \|x_n\|$, $n = 0, 1, 2, \dots$.

4) There do not exist $f \in X^*$, $x_0 \in X$ and a sequence $\{g_n\}_{n\geq 1}$ in G without a weakly convergent subsequence such that $\|f\| = 1$, $f|_G = 0$ and $f(x_0) = \|x_0\| = \|x_0 - g_n\|$, $n = 1, 2, \dots$.

Proof. By Jame's Theorem ([3]), (1) and (2) are equivalent. If we replace compactness by weakly compactness and convergence by weakly convergence in the proof [7; Theorem 2.5], then (1), (3) and (4) are equivalent.■

3. Quasi-Chebyshev Subspaces

In this section we characterize the relation between upper semi-continuity and quasi-Chebyshevity.

3.1. Theorem. *Let X be a Banach space and let G be a proximinal subspace of X with codimension one, then the following are equivalent:*

1) G is quasi-Chebyshev in X.

2) Each sequence $\{y_n\}_{n\geq 1}$ in X with $\|y_n\| = 1$ and $0 \in P_G(y_n)$ ($n = 1, 2, \dots$) has a convergent subsequence.

3) P_G is upper semi-continuous.

Proof. 1)$\Rightarrow$ 2). It has been proved in Lemma 1.3.

2)$\Rightarrow$ 3). Assume that (2) holds. Since $codim(G) = 1$, there exists $f \in X^*$ such that $G = \{y \in X : f(y) = 0\}$. By [11; Lemma 1.2.] we have

$$d(x, G) = \frac{|f(x)|}{\|f\|}, \ (x \in X \backslash G).$$

Let N be an arbitrary closed subset of $\hat{G}$. We shall show that the set $B = N + G$ is closed in X.

For this, let $x \in \overline{B}$. Then there exists a sequence $\{x_n\}_{n\geq 1}$ in B such that $x_n \longrightarrow x$. It follows that there exists a sequence $\{y_n\}_{n\geq 1}$ in N such that $x_n - y_n \in G$ ($n = 1, 2, \dots$). Hence $f(x_n) = f(y_n)$ for all $n \geq 1$ and $\|y_n\| = d(y_n, G)$ ($n = 1, 2, \dots$). Let

$$w_n = \frac{y_n}{\|y_n\|} \ , \ (n = 1, 2, 3, \dots).$$

Then $\|w_n\| = 1$ and $0 \in P_G(w_n)$. Now, by hypothesis $\{w_n\}_{n\geq 1}$ has a convergent subsequence $\{w_{n_k}\}_{k\geq 1}$ with limit w_0. We have $\|y_{n_k}\| = \dfrac{|f(x_{n_k})|}{\|f\|}$ and $\{x_{n_k}\}$ converges to x. Therefore

$$\lim_{k\to\infty} \|y_{n_k}\| = \frac{|f(x)|}{\|f\|},$$

and $y_{n_k} = w_{n_k}\|y_{n_k}\| \longrightarrow w_0\dfrac{|f(x)|}{\|f\|} =: y_0$. Since N is a closed set, $y_0 \in N$ and hence $x = y_0 + (x - y_0) \in N + G = B$. Thus, P_G is upper semi-continuous.

3)$\Rightarrow$ 1). Let $x \in X\backslash G$ and let $\{y_n\}_{n\geq 1}$ be an arbitrary sequence in $P_G(x)$. Then $\{x - y_n\}_{n\geq 1}$ is a sequence in $\hat{G}$. Since

$$\|x - y_n\| = d(x - y_n, G) = d(x, G) \leq \|x\|,$$

for all $n \geq 1$. It follows that $\{x - y_n\}_{n\geq 1}$ is a bounded sequence in $\hat{G}$. But, by [9, Theorem 4.25], $\hat{G}$ is boundedly compact. Therefore, there exists a convergent subsequence $\{x - y_{n_k}\}_{k\geq 1}$ with limit in $\hat{G}$.

Now, since $y_{n_k} = x - (x - y_{n_k})$ and $\|x - y_{n_k}\| = d(x, G)$ for all $k \geq 1$, $\{y_{n_k}\}_{k\geq 1}$ converges to $x_0 \in P_G(x)$. Thus, $P_G(x)$ is compact and hence G is a quasi-Chebyshev subspace of X.■

3.2. Theorem. *Let X be a Banach space, G be a proximinal subspace of X, and P_G be upper semi-continuous. Then G is a quasi-Chebyshev subspace of X.*

Proof. Suppose that P_G is upper semi-continuous on X. Then P_G is upper semi-continuous on every $Y_x = G \oplus \langle x\rangle$ $(x \in X\backslash G)$. Since $codim(G) = 1$ in each Y_x $(x \in X\backslash G)$, by Theorem 3.1 (the implication 3)$\Rightarrow$ 2)) each sequence $\{y_n\}_{n\geq 1}$ in Y_x with $\|y_n\| = 1$ and $0 \in P_G(y_n)$ $(n = 1, 2, \dots)$ has a convergent subsequence and hence by Lemma 1.4, G is quasi-Chebyshev in each Y_x $(x \in X\backslash G)$. But $X = \bigcup_{x\in X\backslash G} Y_x$. It is clear that G is quasi-Chebyshev in X.■

3.3. Corollary. *Let X be a Banach space and let G be a proximinal subspace of X with codimension one. If G is pseudo-Chebyshev, then P_G is upper semi-continuous.*

Proof. This is an immediate consequence of Theorem 3.1 and that every pseudo-Chebyshev subspace is quasi-Chebyshev.■

Now, we shall give an example in which the converse of Corollary 3.3 is not true.

3.4. Example. Let $X = \ell^1 \oplus \langle y_0 \rangle$, $\|y_0\| = 1$ and define a norm on X by

$$\|x + y\| = \sum_{n=1}^{\infty} [|c_n| \vee (2^{-n}\|y\|)] < \infty,$$

where $\sum_{n=1}^{\infty} c_n e_n = x \in W = \ell^1$, $y \in W_0 = \langle y_0 \rangle$ and $a \vee b = \max(a, b)$.

It is clear that $\| \cdot \|$ is a norm on X, and X is a Banach space with respect to this norm. In [7; Example 2.2] it is shown that $W = \ell^1$ is not a pseudo-Chebyshev subspace of X, but is a quasi-Chebyshev subspace of X. Since $codim W = 1$, by Theorem 3.1, P_G is upper semi-continuous.■

By [8; Theorem 4], it follows that if $codim(G) > 1$ and G is pseudo-Chebyshev or G is quasi-Chebyshev, then P_G is not upper semi-continuous, and hence the converse of Theorem 3.2 is not true.

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CHARACTERIZATIONS BASED ON LOWER VARIANCE BOUNDS VIA CHERNOFF-TYPE INEQUALITIES

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Abstract. In this paper, we establish results related to lower bound for the variance as an extended version of Chernoff-type inequalities. This subsume several of the earlier specialized results in the literature and has linked with the lower bound for the variance in Cacoullos & Papathanasiou (1997).

[1]

1. Introduction

There is an extensive literature dealing with upper and lower bounds for the variance of a function of a random variable via Chernoff (1981), which gives a bound for the variance of an absolutely continuous function (w.r.t. Lebesgue measure) of a normal random variable. Chen (1982), Cacoullos (1982) and Klaassen (1985) obtained variations of the inequality relative to other distributions and also gave the corresponding lower bounds. Several papers have appeared on modified versions or variants of the Chernoff inequality such as Borovkov & Utev (1983),

[1] **Keywords:** Chernoff-type Inequalities, Variance bounds, Characterization, Upper bounds, Lower bounds.

Cacoullos & Papathanasiou (1985,1989), Koicheva (1993), Prakasa Rao & Sreehari (1986,1987), Srivastava & Sreehari (1987,1990), Prakasa Rao (1992), Prakasa Rao & Sreehari (1997), Purkayastha & Bhandari (1990), Cacoullos & Papathanasiou (1992) and Hwang & Sheu (1987).

It is natural to ask whether one could extend the aforementioned characterizations, involving exclusively purely absolutely continuous or exclusively purely discrete distributions to a more general format where a distribution has a singular continuous component or it is a non-trivial mixture of a purely discrete distribution and a purely continuous distribution.

We establish a general format for characterizations in this area which is an extended version of that published by Alharbi & Shanbhag (1996) and Mohtashami Borzadaran & Shanbhag (1998); these subsumes several of the earlier specialized results in the literature. Cacoullos & Papathanasiou (1995) published a paper about the generalization of the covariance identity and related characterizations. Cacoullos & Papathanasiou (1997) found lower and upper bounds for the variance of a real-valued function of an r.v. based on a generalization of the covariance identity. We shall establish a link between these results and the results of Alharbi & Shanbhag (1996) and our results. The results in Cacoullos & Papathanasiou (1995,1997) subsume most of the earlier characterizations based on variance bounds. Obviously our findings now will reveal that the results in Cacoullos & Papathanasiou (1995,1997) are corollaries to those in our results.

2. Main Results

We establish here an extended version of Chernoff inequality where lower variance bound in Alharbi& Shanbhag (1996) is a corollary of ours via the following theorems :

Theorem 2.1. *Let F^* be a non-constant Lebesgue-Stieltjes measure function on $\Re$ and ν_{F^*} be the measure on the Borel $\sigma-$ field of $\Re$ determined by it, and let h^* and Z be Borel measurable functions. Let X be an r.v. with df F such that $h^*(X)$ is integrable with $\mu^* = E[h^*(X)]$ and $E(|Z(X)|I_{\{X\in(a,b)\}}) < \infty$ for every $-\infty < a < b < \infty$ and satisfying the condition that $\liminf_{x\to\infty}(h^*(x) - \mu^*) > 0$ if the right extremity of F equals ∞, and the condition that $\liminf_{x\to-\infty}(\mu^* - h^*(x)) > 0$ if the left extremity of F equals $-\infty$. Further let τ be the class of real-valued absolutely continuous functions g with Radon-Nikodym derivative g' w.r.t. the measure ν_{F^*} (i.e. such that $g(b) - g(a) = \int_{(a,b]} g'(x)d\nu_{F^*}(x)$ for all a and b with $a < b$). Then, we have the condition*

$$Cov\{g(X),\ h^*(X)\} = E\{Z(X)g'(X)\}, \tag{1}$$

met for all $g \in \tau$ with $E\big(\mid Z(X)g'(X) \mid \big) < \infty$, if and only if,

$$Z(x)dF(x) = \{\int_{[x,\infty)}[h^*(z) - \mu^*]dF(z)\}d\nu_{F^*}(x),\ x \in \Re. \tag{2}$$

(We read (1) as the condition where the left hand side of the identity is well defined and equals the right hand side of the identity.)

Proof: The “ if ” part can be proved via an extended version of the argument as in Alharbi & Shanbhag (1996) by applying Fubini's theorem as follows :
(2) implies that if $E(|Z(X)g'(X)|) < \infty$, then, for any $a \in \Re$,

$$\begin{aligned} E\{Z(X)g'(X)\} &= \int_{\Re} g'(x)\{\int_{[x,\infty)}[h^*(z) - \mu^*]dF(z)\}d\nu_{F^*}(x) \\ &= \int_{(a,\infty)} g'(x)\{\int_{[x,\infty)}[h^*(z) - \mu^*]dF(z)\}d\nu_{F^*}(x) \\ &+ \int_{(-\infty,a]} g'(x)\{\int_{(-\infty,x)}[\mu^* - h^*(z)]dF(z)\}d\nu_{F^*}(x) \end{aligned} \tag{3}$$

Note that for sufficiently large positive number k, we have, in view of the lim inf conditions concerning h^*,

$$\begin{aligned}&\int_{[k,\infty)} |g'(x)|\big(\int_{[x,\infty)} |h^*(z)-\mu^*|dF(z)\big)d\nu_{F^*}(x)\\ &=\int_{[k,\infty)} |g'(x)|\big(\int_{[x,\infty)} [h^*(z)-\mu^*]dF(z)\big)d\nu_{F^*}(x)<\infty \qquad (4)\end{aligned}$$

and

$$\begin{aligned}&\int_{(-\infty,-k]} |g'(x)|\big(\int_{(-\infty,x)} |\mu^*-h^*(z)|dF(z)\big)d\nu_{F^*}(x)\\ &=\int_{(-\infty,-k]} |g'(x)|\big(\int_{(-\infty,x)} [\mu^*-h^*(z)]dF(z)\big)d\nu_{F^*}(x)<\infty. \qquad (5)\end{aligned}$$

Also, in view of the absolute continuity of g, we get g' to be ν_{F^*}-integrable on $(-k,k)$ and we get for $k>|a|$

$$\begin{aligned}&\int_{(a,k)} |g'(x)|\big(\int_{[x,\infty)} |h^*(z)-\mu^*|dF(z)\big)d\nu_{F^*}(x)\\ &+\int_{(-k,a]} |g'(x)|\big(\int_{(-\infty,x)} |h^*(z)-\mu^*|dF(z)\big)d\nu_{F^*}(x)\\ &\le \big(\int_{(-k,k)} |g'(x)|d\nu_{F^*}(x)\big)E\big(|h^*(z)-\mu^*|\big)\\ &<\infty\end{aligned} \qquad (6)$$

In view of (4), (5) and (6), it flows that the right hand side of (3) with $|g'(x)|$ in place of $g'(x)$ and $|h^*(z)-\mu^*|$ in place of $h^*(z)-\mu^*$ and $\mu^*-h^*(z)$ is finite. Consequently, we can apply Fubini's theorem to the right hand side of (3) to get that it equals

$$\begin{aligned}&\int_{(a,\infty)} \big(g(z)-g(a)\big)[h^*(z)-\mu^*]dF(z)\\ &+\int_{(-\infty,a]} \big(g(a)-g(z)\big)[\mu^*-h^*(z)]dF(z),\end{aligned} \qquad (7)$$

which in turn, equals

$$E\{\big(g(X)-g(a)\big)\big(h^*(X)-\mu^*\big)\};$$

in view of the lim inf conditions concerning h^* and the integrability of $\big(g(X)-g(a)\big)\big(h^*(X)-\mu^*\big)$ it follows that g is integrable and the expectation that we have above equals $Cov\{h^*(X), g(X)\}$. The " only if " part could be proved as follows :
We have

$$E\{Z(X)g'(X)\} = \int_{\Re} g'(x)Z(x)dF(x)$$

and with $a \in \Re$,

$$\begin{aligned}
Cov\{g(X), h^*(X)\} &= E\{g(X)[h^*(X) - E\big(h^*(X)\big)]\} \\
&= \int_{\Re} g(x)[h^*(x)-\mu^*]dF(x) \\
&= \int_{\Re} \big(g(a) + \int_{(a,x]} g'(y)d\nu_{F^*}(y)\big)[h^*(x)-\mu^*]dF(x) \\
&= \int_{\Re} \int_{(a,x]} g'(y)d\nu_{F^*}(y)[h^*(x)-\mu^*]dF(x) \\
&= \int_{(a,\infty)} \int_{(a,x]} g'(y)d\nu_{F^*}(y)[h^*(x)-\mu^*]dF(x) \\
&- \int_{(-\infty,a]} \int_{(x,a]} g'(y)d\nu_{F^*}(y)[h^*(x)-\mu^*]dF(x) \\
&= \int_{(a,\infty)} \big(\int_{[y,\infty)} [h^*(x)-\mu^*]dF(x)\big)g'(y)d\nu_{F^*}(y) \\
&+ \int_{(-\infty,a]} \big(\int_{[y,\infty)} [h^*(x)-\mu^*]dF(x)\big)g'(y)d\nu_{F^*}(y) \\
&= \int_{\Re} \big(\int_{[y,\infty)} [h^*(x)-\mu^*]dF(x)\big)g'(y)d\nu_{F^*}(y),
\end{aligned}$$

Let $-\infty < a < b < \infty$ and let g be absolutely continuous w.r.t. ν_{F^*} such that

$$g'(x) = \begin{cases} 0 & \text{if } x \notin (a,b) \\ 1 & \text{if } x \in (a,b). \end{cases}$$

Hence

$$\int_{(a,b)} Z(x)dF(x) = \int_{(a,b)} \big(\int_{[x,\infty)} [h^*(y) - \mu^*]dF(y)\big) d\nu_{F^*}(x),$$

for all arbitrary $a,\ b > 0$. This implies that

$$\big(\int_{[x,\infty)} \big(h^*(t) - \mu^*]\big)dF(t)\big)d\nu_{F^*}(x) = Z(x)dF(x),$$

which is (2).□

Theorem 2.2. *Let $X,\ g,\ \tau,\ Z$ and h^* be defined as in Theorem 2.1, but additionally with h^* absolutely continuous w.r.t. ν_{F^*} and $h^*(X)$ as nondegenerate square integrable satisfying*

$$Var\{h^*(X)\} = E\big(Z(X)h^{*\prime}(X)\big). \tag{8}$$

Furthermore, let τ^* be the set of $g \in \tau$ for which $g(X)$ is square integrable and $E\{Z(X)g'(X)\}$ is defined and nonzero. Then

$$\inf_{g\in\tau^*} \frac{Var[g(X)]Var[h^*(X)]}{E^2\{Z(X)g'(X)\}} = 1, \tag{9}$$

if and only if (2) holds.

Proof: We shall first establish the " if " part; note that (9) is equivalent to

$$Var\{g(X)\}Var\{h^*(X)\} \geq E^2\{Z(X)g'(X)\},\ g \in \tau^*, \tag{10}$$

on noting that the equality in (10) holds for some g. Clearly, if we assume (2), we have

$$E\{Z(X)g'(X)\} = Cov[g(X), h^*(X)], \tag{11}$$

as seen in Theorem 2.1. Note now that the equality in (10) holds if $g(.) = h^*(.)$. Hence, under the stated assumptions,

$$\begin{aligned} E^2\{Z(X)g'(X)\} &= \{Cov[g(X), h^*(X)]\}^2 \\ &\leq Var[g(X)]Var[h^*(X)], \end{aligned} \tag{12}$$

with equality in (12) if $g = h^*$. This establishes the " if " part of the theorem. The " only if " part may be proved by extending the method of Alharbi & Shanbhag (1996) as follows :

Let (a, b) be a bounded open interval and

$$k(x) = \begin{cases} 0 & \text{if } x \notin (a,b) \\ 1 & \text{if } x \in (a,b), \end{cases}$$

where $-\infty < a < b < \infty$. For any real θ, we define

$$g(x) = h^*(x) - \mu^* + \theta \int_{(-\infty,x]} k(y)d\nu_{F^*}(y), \quad x \in \Re.$$

Clearly, in view of the relations (8) and (9)

$$\begin{aligned} & Var\{h^*(X)\} + \theta^2 Var\{\int_{(-\infty,X]} k(y)d\nu_{F^*}(y)\} \\ + \;& 2\theta Cov\{h^*(X) - \mu^*, \int_{(-\infty,X]} k(y)d\nu_{F^*}(y)\} \\ \geq \;& E[Z(X)h^{*\prime}(X))] + \frac{1}{Var[h^*(X)]}\theta^2 E^2[Z(X)k(X)] \\ + \;& 2\theta E[Z(X)k(X)]. \end{aligned} \tag{13}$$

We see that

$$\begin{aligned} & \theta^2\big(Var\{\int_{(-\infty,X]} k(y)d\nu_{F^*}(y)\} - \frac{1}{Var[h^*(X)]}\{E^2[Z(X)k(X)]\}\big) \\ + \;& 2\theta\big(Cov\{h^*(X) - \mu^*, \int_{(-\infty,X]} k(y)d\nu_{F^*}(y)\} \\ - \;& E[Z(X)k(X)]\big) \geq 0. \end{aligned} \tag{14}$$

Because (14) holds for all θ, it implies

$$Cov\{h^*(X) - \mu^*, \int_{(-\infty,X]} k(y) d\nu_{F^*}(y)\} = E[Z(X)k(X)].$$

In view of Fubini's theorem

$$\int_{(a,b)} Z(x)dF(x) = \int_{(a,b)} \{\int_{[x,\infty)} [h^*(z) - \mu^*]dF(z)\} d\nu_{F^*}(x),$$

which implies (2). Hence we have the theorem.□

Corollary 2.3. *Let F^*, Z and τ be as in Theorem 2.1, and X be an r.v. with df F such that $F^*(X)$ is integrable with $\mu^* = E(F^*(X))$ and $E(|Z(X)|I_{\{X\in(a,b)\}}) < \infty$ for every $-\infty < a < b < \infty$. Then*

$$Cov\{g(X),\ F^*(X)\} = E\{Z(X)g'(X)\}, \tag{15}$$

for all $g \in \tau$ such that $E\{Z(X) \mid g'(X) \mid\} < \infty$, if and only if (2) holds.

Proof: The corollary follows easily from Theorem 2.1 on taking $h^*(.) = F^*(.)$ and noting that the assumptions of the theorem are met. □

Corollary 2.4. (Alharbi & Shanbhag (1996)). *Let X, h^*, Z and τ^* be as defined in Theorem 2.2, but with $h^* = F^*$. Then*

$$\inf_{g\in\tau^*} \frac{Var[g(X)]Var[F^*(X)]}{E^2\{Z(X)g'(X)\}} = 1, \tag{16}$$

if and only if (2).

Proof: The corollary follows immediately from Theorem 2.2 on taking $h^* = F^*$.□

Cacoullos & Papathanasiou (1995) generalized the covariance identity for univariate random variables and used them to obtain several characterizations. Some lower and upper variance bounds were derived by them using a generalization based on the covariance identity, appearing in Cacoullos & Papthanasiou (1997). The paper provides one with a further tool for characterizations in terms of Z (instead of w met earlier in connection with characterizations based on the Chernoff inequality).

Remark 2.5. Let $h^*(x) = x$, then (1) in Theorem 2.1 reduces to

$$Cov\big(X, g(X)\big) = E\big(Z(X)g'(X)\big),$$

for all $g \in \tau$ and

$$Z(x)dF(x) = \Big(\int_{[x,\infty)} (z-\mu)dF(z)\Big)d\nu_{F^*}(x), \tag{17}$$

in place of (2) where $\mu = E(X)$. In this case $F^*(x) = x$ and $h(x) = x$ implies that

$$Z(x)f(x) = \int_{[x,\infty)} (z-\mu)f(z)dz,$$

in place of (17) where f is the density of F w.r.t. Lebesgue measure. Also, in this case, $F^*(x) = [x]$, $x \in \Re$ implies

$$Z(x)f(x) = \sum_{\{y: y \geq x,\ y \in Z\}} (y-\mu)f(y),$$

in place of (17).

Remark 2.6. Using the operator $\Delta_\beta g(.) = \frac{g(.+\beta)-g(.)}{\beta}$, $\beta > 0$, we can obtain extended versions of certain results appearing in Cacoullos & Papathansiou (1995, 1997). The results corresponding to the lattice case involving the operator Δ_β obviously are corollaries to our general results.

Corollary 2.7. *Let h and Z be absolutely continuous Borel measurable functions (w.r.t. Lebesgue measure) and let X be an r.v. with df F such that $h(X)$ is integrable with $\mu = E[h(X)]$ and $E\{|Z(X)|I_{\{X\in(a,b)\}}) <$*

∞ for every $-\infty < a < b < \infty$ and satisfying the condition that $\liminf_{x\to\infty}(h(x) - \mu) > 0$ if the right extremity of F equals ∞, and the condition that $\liminf_{x\to-\infty}(\mu - h(x)) > 0$ if the left extremity of F equals $-\infty$. Further let τ be the class of real-valued absolutely continuous functions g with Radon-Nikodym derivative g' w.r.t. Lebesgue measure. Then, we have the condition

$$Cov\{g(X),\ h(X)\} = E\{Z(X)g'(X)\}, \tag{18}$$

met for all $g \in \tau$ with $E\big(\mid Z(X)g'(X) \mid\big) < \infty$, if and only if

$$Z(x)dF(x) = \{\int_{[x,\infty)} [h(z) - \mu]dF(z)\}dx,\ x \in \Re. \tag{19}$$

Corollary 2.8. *Let $X,\ \tau,\ Z,$ and h be as defined in Corollary 2.7, but additionally with h absolutely continuous w.r.t. Lebesgue measure, τ^* as a subset of $g \in \tau$ for which $g(X)$ is square integrable, $Z(X)g'(X)$ is integrable with $E\{Z(X)g'(X)\} \neq 0$ and $h^2(X)$ integrable and $V[h(X)] = E[Z(X)h'(X)]$. Then*

$$Var[g(X)] \geq \frac{E^2[Z(X)g'(X)]}{E[Z(X)h'(X)]},\ g \in \tau^*, \tag{20}$$

if and only if

$$Z(x)dF(x) = \{\int_{[x,\infty)} [h(y) - E(h(X))]dF(y)\}dx,\ x \in \Re, \tag{21}$$

for all $g \in \tau^*$.
Equality in (20) holds if $g(.) = c_1h(.) + c_2$.

Proof: The result follows from Theorem 2.2 on taking $F^*(x) = x,\ x \in \Re$.

Corollary 2.9. *Let F^* be a non-constant Lebesgue-Stieltjes measure function on $\Re$ and ν_{F^*} be the measure on the Borel $\sigma-$ field of $\Re$ determined by it, and let X be an r.v. with df F such that F is absolutely continuous w.r.t. ν_{F^*} with Radon-Nikodym derivative f w.r.t. ν_{F^*} and $f(x)$ tends to zero as $x \to \infty$. Also, let f be absolutely continuous w.r.t. ν_{F^*} with Radon-Nikodym derivative f' w.r.t. ν_{F^*} such that $E[h^*(X)] = 0$ and $\liminf_{x\to\infty} h^*(x) > 0$ if the right extremity of F equals to ∞, and $\liminf_{x\to-\infty} h^*(x) < 0$ if the left extremity of F equals to $-\infty$, where $h^*(x) = -\frac{f'(x)}{f(x)}$. Further let τ be the class of real-valued absolutely continuous functions g with Radon-Nikodym derivative g' w.r.t. the measure ν_{F^*} (i.e. such that $g(b) - g(a) = \int_{(a,b]} g'(x)d\nu_{F^*}(x)$ for all a and b with $a < b$). Then, we have the condition*

$$Cov\{g(X-),\ h^*(X)\} = E\big(g'(X)\big), \tag{22}$$

met for all g with $E\big(\mid g'(X) \mid\big) < \infty$.

Proof: We have

$$\begin{aligned}
E\{g'(X)\} &= \int_{\Re} g'(x)dF(x) \\
&= \int_{\Re} g'(x)f(x)d\nu_{F^*}(x) \\
&= \int_{\Re} g'(x)\{\int_{(x,\infty)}[h^*(z)]dF(z)\}d\nu_{F^*}(x) \\
&= \int_{(a,\infty)} g'(x)\{\int_{(x,\infty)}[h^*(z)]dF(z)\}d\nu_{F^*}(x) \\
&+ \int_{(-\infty,a]} g'(x)\{\int_{(-\infty,x]}[-h^*(z)]dF(z)\}d\nu_{F^*}(x) \\
&= \int_{(a,\infty)} \big(g(z-) - g(a)\big)[h^*(z)]dF(z) \\
&+ \int_{(-\infty,a]} \big(g(a) - g(z-)\big)[-h^*(z)]dF(z) \\
&= Cov\{h^*(X), g(X-)\}
\end{aligned} \tag{23}$$

Hence completed the proof.$\Box$

Remark 2.10. If F^* is a non-atomic measure, then the Corollary 2.9 is valid with $Cov\{g(X),\ -\frac{f'(X)}{f(X)}\} = E\big(g'(X)\big)$ in place of (22).

3. Further Possible Work

Chen (1982), Cacoullos & Papathanasiou (1989) and Prakasa Rao & Sreehari (1986) set-up characterization results based on versions of the Chernoff-type inequalities for multivariate normal distribution. Also, Prakasa Rao (1993) obtained result related to probability bounds, multivariate normal distribution and integro-differential inequality for a random vector.

An extension of the general results that are obtained here is applicable to a multivariate set-up. The ideas are given for the future work of the lower variance bounds as follows :

- A Lebesgue-Stieltjes measure on $\Re^n$ is a measure μ on $\mathcal{B}(\Re^n)$ such that $\mu(I) < \infty$ for each bounded interval I. Let μ be finite measure, define

$$F(x_1, x_2, ..., x_n) = \mu\Big(\big(\{\omega = (\omega_1, \omega_2, ..., \omega_n\big) \in \Re^n : \ \omega_i \leq x_i, \ i = 1, 2, ..., n\}$$

This will turn out that $\mu(a, b] = F(b) - F(a)$ for $n \geq 2$ is not correct. Hence, introduce the difference operator Δ (for more details see Ash (1972), pp. 27) as follows :

If $G : \Re^n \to \Re$, then,

$$\Delta_{b_i a_i} G(x_1, x_2, .., x_n) = G(x_1, .., x_{i-1}, b_i, x_{i+1}, .., x_n) - G(x_1, .., x_{i-1}, a_i, x_{i+1}, .., x_n).$$

The following theorem gives us the idea for extending main theorems of this note to a multivariate set-up by using Δ operator. (Note that the best way is that first of all, for $n = 2$ obtain the result and then extend it to multivariate case.)

Theorem 3.1. (Ash (1972), pp. 28)

Let μ be a finite measure on $\mathcal{B}(\Re^n)$ and define $F(x) = \mu(-\infty, x]$. If $a \leq b$, then

$$\begin{aligned} \mu(a,b] &= \Delta_{b_1 a_1}\Delta_{b_2 a_2}...\Delta_{b_n a_n}F(x_1, x_2, ..., x_n) \\ &= F_0 - F_1 + F_2 - F_3 + ... + (-1)^n F_n, \end{aligned}$$

where F_i is the sum of all $\binom{n}{i}$ terms of the form $F(c_1, c_2, ..., c_n)$ with $c_k = a_k$ for exactly i integer in $\{1, 2, ..., n\}$ and $c_k = b_k$ for the remaining $n - i$ integer.

Chou (1988) derived an identity as a property of exponential family in $\Re^n$ (see page 130-132 of the Chou's (1988)). We can have the following result using the argument in the earlier paper :

- Theorems 2.1 and 2.2 are extendable to a multivariate version by using a version of C-S inequality (see Cacoullos (1989), pp. 242) and using the argument similar to the Theorem 2.1 in Chou (1988).

Also, the following result can be achieved as a representation in terms of $Z(.)$.

- In view of the argument of Theorem 2.1, giving a criterion under which any distribution F satisfying (2) can be identified by $(h^*,\ F^*,\ \mu^*,\ Z)$.

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SCHUR PROPERTY OF THE DUAL CLOSED SUBSPACES OF COMPACT OPERATORS

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Abstract. Following a work of S. W. Brown [1] and A. Ülger [8] we will prove that for a closed subspace $\mathcal{M} \subseteq K(l_p, l_q)$, $\mathcal{M}^*$ has the Schur property if and only if all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\} \subseteq l_q$ and $\widetilde{\mathcal{M}_1}(y^*) = \{T^*y^* : T \in \mathcal{M}_1\} \subseteq l_{p'}$ are relatively norm compact, where $1 < p \leq q < \infty$, $x \in l_p$, $y^* \in l_{q'}$ and p' is the conjugate number of p. Also for closed subspace $\mathcal{M}$ of either $K(l_p, c_0)$ with $1 < p < \infty$ or $K(c_0)$, we will prove that relative compactness of all point evaluations is also sufficient for the Schur property of $\mathcal{M}^*$.

1. Introduction.

A Banach space X has the Schur property if every weakly convergent sequence in X converges in norm. By Schur's theorem, the sequence space l_1 has the Schur property. In more general, Carne, Cole and Gamelin in [2] have proved that $L^1(\mu)$ has the Schur property if and only if μ is an atomic measure.

Unfortunately, the most of the well-known Banach spaces does not have the Schur property. For example all infinite dimensional reflexive Banach spaces does not have the Schur property.

By Rosenthal's l_1- theorem [4], any infinite dimensional Banach space X with the Schur property, contains a copy of l_1. But if X is a dual space with the Schur property, then its predual contains no copy of l_1 [3].
If X and Y are Banach spaces such that X^* and Y do not contain l_1 and either X^{**} or Y^* has the Radon-Nikodym property (in particular, if X and Y are reflexive), then the Banach space $K(X,Y)$, of all compact operators between X and Y, contains no copy of l_1. This shows that $K(X,Y)$ contains no infinite dimensional closed subspace with the Schur property. But the dual of them can possess this property. So it is natural to ask for which Banach spaces X and Y and which class of closed subspaces $\mathcal{M}$ of $K(X,Y)$, $\mathcal{M}^*$ have the Schur property.

In 1995, Scott W. Brown [1] has proved that if $\mathcal{M}$ is a closed subspace of $K(H)$, of all compact operators on a Hilbert space H, such that all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1 = \text{the unit ball of } \mathcal{M}\}$ and $\widetilde{\mathcal{M}}_1(x) = \{T^*x : T \in \mathcal{M}_1\}$ are relatively norm compact in H, then $\mathcal{M}^*$ has the Schur property. Conversely, in 1997, A. Ülger [8] by proving that relatively norm compactness of all point evaluations is also a necessary condition for the Schur property of $\mathcal{M}^*$, has characterized all closed subspaces of $K(H)$ whose duals have the Schur property:

Theorem 1 *(S. W. Brown [1] and A. Ülger [8]). If H is a Hilbert space and $\mathcal{M}$ is a closed subspace of $K(H)$, then $\mathcal{M}^*$ has the Schur property if and only if for each $x \in H$, the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(x^*)$*

are relatively norm compact in H.

In the following, we will prove that the same conclusion is valid, where $\mathcal{M}$ is a closed subspace of either $K(l_p, l_q)$ with $1 < p \leq q < \infty$ or $K(l_p, c_0)$ with $1 < p < \infty$ or $K(c_0)$. Note that if $p > q$, by Pitt's theorem, $K(l_p, l_q)$ is reflexive and so has no infinite dimesional closed subspace $\mathcal{M}$ such that either $\mathcal{M}$ or $\mathcal{M}^*$ has the Schur property.
In the following (e_n) is the standard basis for both c_0 and all l_p, $1 < p < \infty$ and P_V is the projection onto complemented closed subspace V of the related space. Also X_1 is the closed unit ball of arbitrary Banach space X. We denote the conjugate number of p by p'. For the proof of one of the main theorems we will first prove three lemmas:

Lemma 2. *Let $K_1, ..., K_n \in K(l_p, l_q)$ and $\epsilon > 0$ be given. Then there are integers m_0 and n_0 such that*

$$\|P_{W'} K_i\| \leq \epsilon \ \& \ \|K_i P_{V'}\| \leq \epsilon \ , \ \ i = 1, 2, ..., n,$$

where $V' = [e_n : n > m_0]$ and $W' = [e_n : n > n_0]$ are closed subspaces of l_p and l_q respectively.

Proof. We may assume, without loss of generality, that $n = 1$ and $K = K_1 \in K(l_p, l_q)$. If $\{y_1, ..., y_l\}$ is an $\epsilon/2$- covering of $K((l_p)_1)$ in l_q and each y_i has a representation $y_i = \sum_{k=1}^{\infty} \alpha_{i_k} e_k$, choose an integer $n_0 > 0$ such that $\| \sum_{k>n_0} \alpha_{i_k} e_k \| = (\sum_{k>n_0} |\alpha_{i_k}|^q)^{1/q} < \epsilon/2$, for all $1 \leq i \leq l$.
Now for each $x \in (l_p)_1$ and suitable $1 \leq i \leq l$,

$$\|P_{W'} Kx\| \leq \|P_{W'}(Kx - y_i)\| + \|P_{W'} y_i\|$$

$$\leq \|Kx - y_i\| + \|\sum_{k>n_0} \alpha_{i_k} e_k\| < \epsilon.$$

This shows that $\|P_{W'}K\| < \epsilon$ where $W' = [e_n : n > n_0]$.
As a corollary, since $K^* : l_{q'} \to l_{p'}$ is compact, we can deduce that there exists an integer m_0 such that $\|PK^*\| < \epsilon$ where P is the canonical projection of $l_{p'}$ onto $[e_n : n > m_0]$ in $l_{p'}$. Set $V' = [e_n : n > m_0]$, as a closed subspace of l_p. Since $P = (P_{V'})^*$ we have

$$\|KP_{V'}\| = \|(P_{V'})^* K^*\| = \|PK^*\| < \epsilon.$$

Lemma 3. *Let m_0 and n_0 be arbitrary integers, $V = [e_1, ..., e_{m_0}] \subseteq l_p$, $W = [e_1, ..., e_{n_0}] \subseteq l_q$ and $\epsilon > 0$ be given. If $\mathcal{M} \subseteq K(l_p, l_q)$ is a closed subspace such that all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\} \subseteq l_q$ and $\widetilde{\mathcal{M}}_1(y^*) = \{T^*y^* : T \in \mathcal{M}_1\} \subseteq l_{p'}$ are relatively compact, then there exists a closed subspace $\mathcal{G}$ of $\mathcal{M}$ of finite codimension such that*

$$\|GP_V\| \leq \epsilon \text{ and } \|P_W G\| \leq \epsilon, \text{ for all } G \in \mathcal{G}_1.$$

Proof. We first construct a norm closed subspace $\mathcal{E}$ of $\mathcal{M}$ of finite codimension such that $\|GP_V\| \leq \epsilon$, for all $G \in \mathcal{E}_1$.
For each integer i, define $\phi_i : \mathcal{M} \to l_q$ by $\phi_i(T) = Te_i$. For each fixed integer $1 \leq i \leq m_0$, if $\{y_1, ..., y_l\}$ is an $\epsilon/3m_0$- covering of $\phi_i(\mathcal{M}_1)$ and each y_j has a representation $y_j = \sum_{n=1}^{\infty} \alpha_{j_k} e_k$, we can choose an integer N such that $\|\sum_{k>N} \alpha_{j_k} e_k\| < \epsilon/3m_0$, for all $1 \leq j \leq l$. Let $H_i = [e_n : n > N]$ in l_q. Then

$$sup\{\|y\| : y \in H_i \cap \phi_i(\mathcal{M}_1)\} \leq \epsilon/m_0.$$

It is easy to check that $\mathcal{E} := \cap_{i=1}^{m_0} \phi_i^{-1}(H_i)$ is norm closed and of finite codimension in $\mathcal{M}$. Let now $G \in \mathcal{E}_1$. Then $\|Ge_i\| \leq \epsilon/m_0$, for all

$1 \leq i \leq m_0$. If $x = \sum_{i=1}^{\infty} \alpha_i e_i \in l_p$ and $\|x\| \leq 1$ then

$$\|GP_V x\| = \|G \sum_{i=1}^{m_0} \alpha_i e_i\| \leq \sum_{i=1}^{m_0} |\alpha_i|.\|Ge_i\| \leq \epsilon.$$

Thus $\|GP_V\| \leq \epsilon$, for all $G \in \mathcal{E}_1$. Similarly, using the relatively compactness of all $\widetilde{\mathcal{M}_1}(y^*)$ in $l_{p'}$, we construct a norm closed subspace $\mathcal{F}$ of $\mathcal{M}$ of finite codimension such that $\|P_W G\| \leq \epsilon$, for all $G \in \mathcal{F}_1$. Now set $\mathcal{G} = \mathcal{E} \cap \mathcal{F}$.

Lemma 4. *For each integers m and n and each operators $T, S \in L(l_p, l_q)$, if $1 \leq p \leq q < \infty$ we have*

$$\|P_W T P_V + P_{W'} S P_{V'}\| \leq max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\},$$

where $V = [e_1, ..., e_m] \subseteq l_p$, $W = [e_1, ..., e_n] \subseteq l_q$ and V' and $W'$are complementary subspaces of V and W in l_p and l_q respectively.

Proof. It is clear that for arbitrary bounded operators $U_1 : X_1 \to Y_1$ and $U_2 : X_2 \to Y_2$, the direct sum operator $U_1 \oplus U_2 : X_1 \oplus_p X_2 \longrightarrow Y_1 \oplus_q Y_2$ has norm equal to $max\{\|U_1\|, \|U_2\|\}$, where $X_1 \oplus_p X_2$ is the l_p-direct sum of X_1 and X_2.

Now for the bounded linear operators $P_W T P_V|_V : V \to W$ (restriction of $P_W T P_V$ to V) and $P_{W'} S P_{V'}|_{V'} : V' \to W'$ we have

$$\begin{aligned}\|P_W T P_V|_V \oplus P_{W'} S P_{V'}|_{V'}\| &= max\{\|P_W T P_V|_V\|, \|P_{W'} S P_{V'}|_{V'}\|\} \\ &\leq max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\}.\end{aligned}$$

Since $V \oplus_p V'$ and $W \oplus_q W'$ are isometrically isomorphic to l_p and l_q respectively, and the operator $P_W T P_V|_V \oplus P_{W'} S P_{V'}|_{V'}$ as an operator from l_p to l_q is equal to $P_W T P_V + P_{W'} S P_{V'}$, the proof is completed.

We are ready to prove the main result:

Theorem 5. *Let $\mathcal{M}$ be a closed subspace of $K(l_p, l_q)$. If all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in l_q and $l_{p'}$ respectively, then $\mathcal{M}^*$ has the Schur property.*

Proof. Let $(\Gamma_i) \subseteq \mathcal{M}^*$ be a normalized weakly null sequence in $\mathcal{M}^*$. Let (ϵ_n) be a sequence of positive numbers such that $\sum n\epsilon_n < \infty$. Suppose that $\Lambda_1 = \Gamma_1$, and choose $K_1 \in \mathcal{M}_1$ such that $< K_1, \Lambda_1 >> 1/3$. Inductively, assume that $\Lambda_1, ..., \Lambda_n \in (\Gamma_i)$ and $K_1, ..., K_n \in \mathcal{M}_1$ have been chosen. To obtain Λ_{n+1} and K_{n+1}, by lemmas 2 and 3 we find finite dimensional subspaces V and W of l_p and l_q respectively, and norm closed subspace $\mathcal{G}$ of finite codimension in $\mathcal{M}$ such that

$$\|K_i P_{V'}\| \leq \epsilon_{n+1} \text{ and } \|P_{W'} K_i\| \leq \epsilon_{n+1} \text{ , for all } i = 1, 2, ..., n,$$

$$\|G P_V\| \leq \epsilon_{n+1} \text{ and } \|P_W G\| \leq \epsilon_{n+1} \text{ , for all } G \in \mathcal{G}_1.$$

By the technique given in the proof of theorem 1.1 of [1], we can choose $\Lambda_{n+1} \in (\Gamma_i)$ and $K_{n+1} \in \mathcal{M}_1$ such that

$$| < K_i, \Lambda_{n+1} > | < 2^{-n-1} \quad \text{for} \quad i = 1, 2, ..., n,$$

$$< K_{n+1}, \Lambda_{n+1} >> 1/3 \text{ and } < K_{n+1}, \Lambda_j >= 0 \quad \text{for} \quad j = 1, 2, ..., n.$$

Also $\|K_{n+1} P_V\| < \epsilon_{n+1}$ and $\|P_W K_{n+1}\| < \epsilon_{n+1}$. These properties yield that

$$\|P_W \sum_{i=1}^{n} K_i P_V - \sum_{i=1}^{n} K_i\| \leq 3n\epsilon_{n+1} \textit{ and } \|P_{W'} K_{n+1} P_{V'} - K_{n+1}\| \leq 3\epsilon_{n+1}.$$

Hence

$$\|\sum_{i=1}^{n+1} K_i\| \leq \|\sum_{i=1}^{n} K_i - P_W \sum_{i=1}^{n} K_i P_V\| + \|K_{n+1} - P_{W'} K_{n+1} P_{V'}\| +$$

$$\|P_W \sum_{i=1}^{n} K_i P_V + P_{W'} K_{n+1} P_{V'}\| \leq 3(n+1)\epsilon_{n+1} + max\{\|\sum_{i=1}^{n} K_i\|, 1\},$$

where the last inequality holds by lemma 4. This shows that the sequence $T_n = \sum_{i=1}^{n} K_i$ is bounded and so has a weak* limit point $T \in \mathcal{M}^{**}$. For each j, choose an integer $n > j$ such that $| < T - T_n, \Lambda_j > \ | < 1/2^j$. Therefore

$$| < T, \Lambda_j > | \geq | \sum_{i=1}^{j} < K_i, \Lambda_j > | - 1/2^j \geq$$

$$| < K_j, \Lambda_j > | - \sum_{i=1}^{j-1} | < K_i, \Lambda_j > | - 1/2^j \geq 1/3 - j/2^j > 1/4,$$

for sufficiently large j. Hence $< T, \Lambda_j >$ and so $< T, \Gamma_j >$ does not tend to zero. This shows that the sequence (Γ_j) does not converge weakly to zero and the proof is completed.

Remark. In 1999 E. Saksman and H. O. Tylli [7], by a different proof have proved that the conclusion of theorem 5 is valid for the closed subspaces $\mathcal{M}$ of $K(l_p)$ with $1 < p < \infty$, which is a particular case of theorem 5. In the following we extend theorem 5 to closed subspaces $\mathcal{M}$ of the non-reflexive case of Banach spaces. Namely in the case of $K(l_p, c_0)$ and $K(c_0)$.

By notice in the proof of theorem 5, one sees that for each Banach spaces X and Y instead of l_p and l_q respectively, and each operator ideal between X and Y , if lemmas 2, 3 and 4 hold, then a similar result of theorem 5 is also valid for that operator ideal. As a corollary, if we repeat the proof of lemmas 2, 3 and 4 for closed subspaces $\mathcal{M}$ of either $K(l_p, c_0)$ with $1 < p < \infty$ or $K(c_0)$, we have the following theorem. We agreement that for the closed subspace $V = [e_1, ..., e_n]$ of c_0, $V \oplus_\infty V'$ is the c_0 direct sum of V and V'.

Theorem 6. *Let $\mathcal{M}$ be a closed subspace of $K(l_p, c_0)$ with $1 < p < \infty$ (resp. $K(c_0)$). If all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in c_0 and $l_{p'}$ (resp. l_1) respectively, then $\mathcal{M}^*$ has the Schur property.*

As another corollary, we will prove that if H is any Hilbert space, then no analogous result of lemma 4 holds for the closed subspaces of the Banach operator ideal $\mathcal{N}(H)$ of all nuclear operators on a Hilbert space H:

Corollary 7. *Let $\mathcal{M} \subseteq \mathcal{N}(H)$ be an infinite dimensional closed linear subspace of $\mathcal{N}(H)$. If all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(x)$ are relatively norm compact in H, then there are finite dimensional subspaces V and W of H and nuclear operators $T, S \in \mathcal{M}$ such that*

$$\|P_W T P_V + P_{W'} S P_{V'}\|_n > max\{\|P_W T P_V\|_n, \|P_{W'} S P_{V'}\|_n\}.$$

Proof. By the definition of nuclear norm, the analogous of lemmas 2 and 3 are valid for $\mathcal{N}(H)$ instead of $K(l_p, l_q)$. If the conclusion is false, then a similar result of lemma 4 is obtained and so $\mathcal{M}^*$ has the Schur property. This shows that $\mathcal{M}$ does not contain any copy of l_1, while $\mathcal{M}$ is non-reflexive. A contradiction with theorem 3 of [5].

The proof of lemma 4 is based on the fact that for each closed subspace $V \subseteq l_p$ of the form $V = [e_1, ..., e_n]$, the Banach space l_p is isometrically

isomorphic to $V \oplus_p V'$. In general, if a Banach space X with Schauder basis (x_n) has the property that for each integer n, there exists $1 \leq p < \infty$ such that $X = V \oplus_p V',\ V = [x_1, ..., x_n]$; then the proof of lemma 4 is valid for $L(X)$ instead of $K(l_p, l_q)$.
In the following, we will prove that the Banach spaces $l_p,\ 1 \leq p < \infty$ and c_0 are the only Banach spaces with the Schauder basis such that the proof of lemma 4 and so the technique of the proof of theorem 5 valid for them:

Theorem 8. *Suppose that X is a Banach space inclined to Schauder basis (x_n), with the property that for each integer n there exists $1 \leq p_n \leq \infty$ such that X is isometrically isomorphic to $V \oplus_{p_n} V'$, where $V = [x_1, ..., x_n]$. Then X is isometrically isomorphic to either l_p, for some $1 \leq p < \infty$ or c_0.*

Proof. As a first step we will prove that all p_n's are independent from choicing of n. If all p_n are less than to infinity, for each integer $n \geq 2$ and each scalars a_{n-1} and a_{n+1} we have

$$\|a_{n-1}x_{n-1} + a_{n+1}x_{n+1}\|^{p_n} = (\|a_{n-1}x_{n-1}\|^{p_{n-1}} + \|a_{n+1}x_{n+1}\|^{p_{n-1}})^{\frac{p_n}{p_{n-1}}}.$$

On the other hand,

$$\|a_{n-1}x_{n-1} + a_{n+1}x_{n+1}\|^{p_n} = \|a_{n-1}x_{n-1}\|^{p_n} + \|a_{n+1}x_{n+1}\|^{p_n}.$$

So if we set $a_{n-1} = 1/\|x_{n-1}\|$ and $a_{n+1} = 1/\|x_{n+1}\|$, then $2^{\frac{p_n}{p_{n-1}}} = 2$. This shows that for all $n \geq 2,\ p_n = p_{n-1}$ and so all p_n are equal. But if for integers $n < k,\ p_n = \infty$ and $p_k < \infty$. we have

$$\|a_n x_n + a_{k+1}x_{k+1}\| = max\{\|a_n x_n\|, \|a_{k+1}x_{k+1}\|\}$$

$$= (\|a_n x_n\|^{p_k} + \|a_{k+1}x_{k+1}\|^{p_k})^{\frac{1}{p_k}},$$

and so $2^{1/p_k} = 1$. This contradiction shows that all p_n are equal to ∞. As the second step, let $x = \sum_{n=1}^{\infty} a_n x_n \in X$ be arbitrary. If all p_n are equal to some $1 \leq p < \infty$, then by induction on n and by hypothesis,

$$\|x\|^p = \|a_1 x_1\|^p + \|\sum_{n=2}^{\infty} a_n x_n\|^p$$

$$= \|a_1 x_1\|^p + \|a_2 x_2\|^p + \|\sum_{n=3}^{\infty} a_n x_n\|^p$$

$$= ... = \sum_{n=1}^{\infty} \|a_n x_n\|^p = \sum_{n=1}^{\infty} |a_n.\|x_n\||^p.$$

Define $T : X \to l_p$ by $T(\sum_{n=1}^{\infty} a_n x_n) = (a_n.\|x_n\|)_{n=1}^{\infty}$. Then T is an isometrically isomorphism from X onto l_p. Finally, if all p_n are equal to ∞, then by a similar method, $\|x\| = \sup_n \|a_n x_n\|$ and so the operator

$$\sum_{n=1}^{\infty} a_n x_n \mapsto (a_n.\|x_n\|)_{n=1}^{\infty}$$

from X onto c_0 is an isometrically isomorphism and the proof is completed.

Remark. Recently, the author in a joint work [6], generalized the main results of this paper to a large class of Banach spaces.

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THE CONNECTION BETWEEN THE TURNING POINTS AND THE DUAL EQUATION

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ABSTRACT. In this paper we will study the dual equation of the Sturm-Liouville with the dirichlet boundary condition. In most differential equations with the boundary conditions it is possible to obtain an the dual equation. In studying the dual equation of Sturm-Liouville problem, necessarily we must to know the numbers of turning points. In this paper at the first outset, if this problem has one turning point, then by a theorem we find the dual equation. In the second case, if this problem has two turning points, we will determine the another dual equation in this case.

1

1. INTRODUCTION

By considering eigenvalues and infinite product of solutions the second order differential equation

$$\frac{d^2W}{d\zeta^2} + \left(\lambda(1-\zeta^2) - \Psi(\zeta)\right) W = 0 \qquad -\infty < a < -1, 1 < b, \zeta \in (-1, b) \tag{1}$$

with boundary conditions

$$W(a) = 0 = W(\zeta) \tag{2}$$

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and initial condition $\frac{\partial W}{\partial \zeta}(\lambda, a) = 1$, where λ is a large positive parameter and the function $\Psi(\zeta)$ is continuous, we first determine the dual equations of (1) in one turning point case. In this regard we discuss the equations (1) and (2), for case $-1 < \zeta < 1$ and case $1 < \zeta < b$.

The solution of equation (1) have several infinite product representation (see [3]). Let $\zeta \in (-1, 0)$, then we have

$$W(\lambda, \zeta) = A(\zeta) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{v_n(\zeta)}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{r_n(\zeta)}\right). \tag{3}$$

In particular, if $\zeta \in (0, 1)$, the differential equation (1) has a solution $W(\lambda, \zeta)$ given by

$$W(\lambda, \zeta) = B(\zeta) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{v_n(\zeta)}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{r_n(\zeta)}\right). \tag{4}$$

Now, if $1 < \zeta < b$, then the infinite product of solution is of the form

$$W(\lambda, \zeta) = C(\zeta) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{v_n(\zeta)}\right) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{r_n(\zeta)}\right). \tag{5}$$

In equation (3), the function $C(\zeta)$ is

$$A(\zeta) = \frac{i^{3/2} \pi f^{1/2}(\zeta) P^{1/2}(-1)}{(a^2 - 1)^{1/4} (\zeta^2 - 1)^{1/4} e^{P(-1)\sqrt{\lambda} + \frac{3\pi i}{4}}} \prod_{n=1}^{\infty} \frac{f^2(\zeta) r_n(\zeta)}{\tilde{j}_n^2} \prod_{n=1}^{\infty} \frac{P^2(-1) v_n(\zeta)}{-\tilde{j}_n^2}$$

where $r_n(\zeta)$ and $v_n(\zeta)$ are eigenvalues of equation (1) with the boundary condition $W(a) = 0 = W(\zeta)$ and $\tilde{j}_n$, $n = 1, 2, 3, \ldots$ be the positive zeros of $J_1'(z)$ and the functions $P(\zeta)$,$f(\zeta)$ are of the form

$$f(\zeta) = \begin{cases} \int_{-1}^{\zeta} (1 - \tau^2)^{1/2} d\tau & -1 < \zeta \leq 0 \\ \frac{\pi}{2} - \int_{\zeta}^{1} (1 - \tau^2)^{1/2} d\tau & 0 < \zeta < 1 \end{cases}$$

$$P(\zeta) = \begin{cases} \int_a^{\zeta} (\tau^2 - 1)^{1/2} d\tau & \zeta \leq 1 \\ \int_a^{-1} (\tau^2 - 1)^{1/2} d\tau + \int_a^{\zeta} (\tau^2 - 1)^{1/2} d\tau & \zeta > 1 \end{cases}.$$

Similar to that, in equation (4) the function $B(\zeta)$ is

$$D(\zeta) = \frac{i^{3/2} f^{1/2}(\zeta) P^{1/2}(-1)}{(a^2-1)^{1/4}(\zeta^2-1)^{1/4} e^{\frac{3\pi i}{4}}} \prod_{n=1}^{\infty} \frac{f^2(\zeta) r_n(\zeta)}{\tilde{j}_n^2} \prod_{n=1}^{\infty} \frac{P^2(-1) v_n(\zeta)}{-\tilde{j}_n^2},$$

and in equation (5) the function $C(\zeta)$

$$C(\zeta) = \frac{4\pi}{(\Gamma(1/2))^2 (a^2-1)^{1/4}(\zeta^2-1)^{1/4}} \prod_{n=1}^{\infty} \frac{f^2(1) r_n(\zeta)}{\tilde{i}_n^2} \prod_{n=1}^{\infty} \frac{P^2(\zeta) v_n(\zeta)}{-\tilde{i}_n^2},$$

$\tilde{i}_n$, $n = 1, 2, 3, \ldots$ be the positive zeros of $J_{-1/2}(z)$. For more details of equation (3) for $\zeta \in (0, 1)$, can be found in [4] and other cases, in [1], he is proved.

2. The dual equation

Let $\zeta \in (-1, b)$, the equation (1) has an infinite number of positive and negative eigenvalues, which we denote by $\{r_n(\zeta)\}$, $\{v_n(\zeta)\}$ respectively. By the implicit function theorem, the functions $r_n(\zeta)$ and $v_n(\zeta)$ are twice continuously differentiable functions. Note that , if $\lambda_n(\zeta)$ (for $n \geq 1$) is eigenvalue, then we have

$$W(r_n(\zeta), \zeta) = 0,$$

$$W(v_n(\zeta), \zeta) = 0,$$

where $r_n(\zeta)$ and $v_n(\zeta)$ are eigenvalues of equation (1)(for $n \geq 1$), therefore we have

$$2\frac{\partial^2 W}{\partial\zeta\partial\lambda} r'_n + \frac{\partial^2 W}{\partial\lambda^2} \times (r'_n)^2 + \frac{\partial W}{\partial\lambda} \times r''_n = 0, \tag{6}$$

$$2\frac{\partial^2 W}{\partial\zeta\partial\lambda} v'_n + \frac{\partial^2 W}{\partial\lambda^2} \times (v'_n)^2 + \frac{\partial W}{\partial\lambda} \times v''_n = 0. \tag{7}$$

Now by using the above results, we want to take the dual equations of (1). The dual equations have different form on subintervals of $(-1, b)$. By taking the differentiation of equation (3) with respect to the variable

λ at the point $(r_n(\zeta),\zeta)$, for $\zeta\in(-1,0)$ the solution of equation (1) has infinite product representation, from (3) we get

$$W(\lambda,\zeta)=A(\zeta)\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{v_n(\zeta)}\right)\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{r_n(\zeta)}\right).$$

we define G_n and H_n by

$$G_n=G_n(r_n(\zeta),\zeta)=\prod_{k\geq 1,k\not{1}}\left(1-\frac{r_n(\zeta)}{r_k(\zeta)}\right),$$

$$H_n=H_n(r_n(\zeta),\zeta)=\prod_{k\geq 1}\left(1-\frac{r_n(\zeta)}{v_k(\zeta)}\right).$$

Therefore, we get

$$\prod_{k\geq 1,k\neq i}\left(1-\frac{r_n(\zeta)}{v_k(\zeta)}\right)=H_n\times\left(1-\frac{r_n(\zeta)}{v_i(\zeta)}\right)^{-1},$$

so, we have

$$\frac{\partial W}{\partial\lambda}(r_n(\zeta),\zeta)=\frac{-A(\zeta)H_nG_n}{r_n(\zeta)}, \tag{8}$$

similarly, for $\frac{\partial^2W}{\partial\lambda^2}(r_n(\zeta),\zeta)$, we write

$$\begin{aligned}\frac{\partial^2W}{\partial\lambda^2}(r_n(\zeta),\zeta) \quad = \quad & \frac{2A(\zeta)H_nG_n}{r_n(\zeta)}\sum_{1\leq i}\frac{1}{v_i(\zeta)-r_n(\zeta)} \\ & +\frac{2A(\zeta)H_nG_n}{r_n(\zeta)}\sum_{1\leq i,i\neq n}\frac{1}{r_i(\zeta)-r_n(\zeta)},\end{aligned} \tag{9}$$

and for $\frac{\partial^2W}{\partial\lambda\partial x}(r_n(\zeta),\zeta)$ we write

$$\frac{\partial^2W}{\partial\lambda\partial x}(r_n(\zeta),\zeta)=\frac{-A'(\zeta)H_nG_n}{r_n(\zeta)}+\frac{A(\zeta)H_nG_nr'_n}{r_n^2(\zeta)} \tag{10}$$

$$\frac{-A(\zeta)H_nG_nr'_n}{r_n(\zeta)}\sum_{1\leq i}\frac{1}{v_i(\zeta)-r_n(\zeta)}-A(\zeta)H_nG_n\sum_{1\leq i}\frac{v'_i}{v_i}\left(v_i(\zeta)-r_n(\zeta)\right)^{-1}$$

$$-A(\zeta)H_nG_n\sum_{1\leq i,i\neq n}\frac{r'_i}{r_i}\left(r_i(\zeta)-r_n(\zeta)\right)^{-1}-\frac{A(\zeta)H_nG_nr'_n}{r_n}\sum_{1\leq,i\neq n}\frac{1}{r_i(\zeta)-r_n(\zeta)}.$$

By substituting (8),(9) and (10) in (6) ,we have

$$r_n'' + \frac{2A'(\zeta)r_n'}{A(\zeta)} + 2r_n r_n' \left\{ \sum_{1\le i, i\neq n} \frac{r_i'}{r_i} (r_i(\zeta) - r_n(\zeta))^{-1} \right. \tag{11}$$

$$\left. + \sum_{1\le i} \frac{v_i'}{v_i} (v_i(\zeta) - r_n(\zeta))^{-1} \right\} - 2\frac{(r_n')^2}{r_n} = 0,$$

similarly for negative eigenvalue $v_n(\zeta)$, we have

$$v_n'' + \frac{2A'(\zeta)v_n'}{A(\zeta)} + 2v_n v_n' \left\{ \sum_{1\le i, i\neq n} \frac{v_i'}{v_i} (v_i(\zeta) - v_n(\zeta))^{-1} \right. \tag{12}$$

$$\left. + \sum_{1\le i} \frac{r_i'}{r_i} (r_i(\zeta) - v_n(\zeta))^{-1} \right\} - 2\frac{(v_n')^2}{v_n} = 0.$$

Dividing the equation (11) by r_n', the equation (12) by v_n' and integrating from ζ up to 0, we obtain

$$r_n'(\zeta) = \frac{r_n^2(\zeta)}{A^2(\zeta)} e^{2T_n(r_n, v_n, \zeta)}, \tag{13}$$

$$v_n'(\zeta) = \frac{v_n^2(\zeta)}{A^2(\zeta)} e^{2T_n(v_n, r_n, \zeta)},$$

where

$$T_n(r_n, v_n, \zeta) = \sum_{i\neq n} \int_\zeta^0 \frac{r_i' r_n}{r_n} (r_i - r_n)^{-1} dt + \sum i \int_\zeta^0 \frac{v_i' r_n}{v_n} (v_i - r_n)^{-1} dt,$$

for $\zeta \in (0, 1)$

$$T_n(r_n, v_n, \zeta) = \sum_{i\neq n} \int_\zeta^1 \frac{r_i' r_n}{r_n} (r_i - r_n)^{-1} dt + \sum i \int_\zeta^1 \frac{v_i' r_n}{v_n} (v_i - r_n)^{-1} dt$$

similarly for $\zeta \in (1, b)$

$$T_n(r_n, v_n, \zeta) = \sum_{i\neq n} \int_\zeta^b \frac{r_i' r_n}{r_n} (r_i - r_n)^{-1} dt + \sum i \int_\zeta^b \frac{v_i' r_n}{v_n} (v_i - r_n)^{-1} dt.$$

In fact we have obtained the following theorems

Theorem 1 *Let $W(\lambda,\zeta)$ be the solution of boundary value problem*

$$\frac{d^2W}{d\zeta^2}+\big(\lambda(\zeta^2-1)-\Psi(\zeta)\big)\,W=0 \qquad -\infty<a<-1,-1\zeta<0,$$

and

$$W(a)=0=W(\zeta) \qquad \frac{\partial W(\lambda,x)}{\partial x}(\lambda,a)=1$$

then for $r_n(\zeta)$, $v_n(\zeta)$ we have

$$r_n'(\zeta)=\frac{r_n^2(\zeta)}{A^2(\zeta)}e^{2T_n(r_n,v_n,\zeta)}, \qquad v_n'(\zeta)=\frac{v_n^2(\zeta)}{A^2(\zeta)}e^{2T_n(v_n,r_n,\zeta)},$$

and for $\zeta\in(0,1)$ we have

$$r_n'(\zeta)=\frac{r_n^2(\zeta)}{B^2(\zeta)}e^{2T_n(r_n,v_n,\zeta)}, \qquad v_n'(\zeta)=\frac{v_n^2(\zeta)}{B^2(\zeta)}e^{2T_n(v_n,r_n,\zeta)},$$

Theorem 2 *Let $W(\lambda,\zeta)$ be the solution of boundary value problem*

$$\frac{d^2W}{d\zeta^2}+\big(\lambda(\zeta^2-1)-\Psi(\zeta)\big)\,W=0 \qquad -\infty<a<-1,1<\zeta<b,$$

and

$$W(a)=0=W(\zeta) \qquad \frac{\partial W(\lambda,x)}{\partial x}(\lambda,a)=1$$

then for $r_n(\zeta)$, $v_n(\zeta)$ we have

$$r_n'(\zeta)=\frac{r_n^2(\zeta)}{C^2(\zeta)}e^{2T_n(r_n,v_n,\zeta)}, \qquad v_n'(\zeta)=\frac{v_n^2(\zeta)}{C^2(\zeta)}e^{2T_n(v_n,r_n,\zeta)}.$$

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OPTIMAL CONTROL FEEDBACK LAW AND NONSMOOTH ANALYSIS

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Abstract. In general, even in very simple examples of optimal control problems, one can not expect the existence of continuous universal feedback laws. In this work we use an approach to end-point cost optimal control based on nonsmooth analysis to construct a discontinuous universal feedback law.

1. Introduction

In control theory, the standard model involves the system

$$\dot{x} = f(t, x, u), \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and locally Lipschitz in the state variable x. Controls are Lebesgue measurable functions $u : \mathbb{R} \to U$, where the control restraint set $U \subseteq \mathbb{R}^m$ is compact. Assume also that f satisfies a linear growth condition; that is, there exist nonnegative constants γ_1 and γ_2 such that

$$\|f(t, x, u)\| \leq \gamma_1 \|x\| + \gamma_2 \quad \forall (t, x, u).$$

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Then for each control function $u(\cdot)$ and each initial phase $(\tau, \alpha) \in \mathbb{R} \times \mathbb{R}^n$ there is a unique solution $x(t) = x(t; \tau, \alpha, u(\cdot))$ on $[\tau, \infty)$. A central issue in control theory is the existence of *feedback control laws* which achieve desired behavior of the control system (1) *universally*; that is, for all initial states or initial phases in a prescribed set. In recent years it has come to be understood that even in very simple examples of the above types of problems, one cannot expect the existence of universal feedback laws k which are continuous, this being the minimal condition for the classical existence theory of ordinary differential equations to apply to (1). This inadequacy of continuous feedback can be illustrated via the following example.

Example 1.1. Consider the following well known "nonholonomic integator" of Brockett [1]. The dynamics $\dot{x} = f(x, u)$ (here time-autonomous; that is, f has no direct t-dependence) are given by

$$\dot{x}_1 = u_1$$

$$\dot{x}_2 = u_2$$

$$\dot{x}_3 = x_1 u_2 - x_2 u_1$$

where controls $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ are required to be valued in the closed unit ball $U = \overline{B}_2$ in $\mathbb{R}^2$. It can be shown by an ad hoc argument that this system is asymptotically controllable. Nevertheless, stabilizability via a locally Lipschitz feedback law $k(x)$ fails here because, as Brockett's arguments show, a necessary condition for this is the existence of $\Delta > 0$ such that

$$\Delta B_3 \subseteq f(\mathbb{R}^3, \overline{B}_2), \tag{2}$$

where B_3 denotes the open unit ball in $\mathbb{R}^3$. It is easy to check that this condition fails for the nonholonomic integrator. In fact, due to a specialization of a result of Ryan (commented on below) which generalizes that of Brockett, no continuous stabilizing feedback exists. For further discussion along these lines, see Sontag and Sussman [20], and Sontag [18], [21].

Unlike the preceding example, it is not always possible to obtain a classical solution to the differential equation

$$\dot{x} = f(t, x, k(t, x)) =: g(t, x)$$

when the feedback k is discontinuous, since the existence theory of ordinary differential equations can break down. One might hope that a remedy for this difficulty is to replace the above differential equation with a differential inclusion $\dot{x} \in G(t, x)$ via either the Krasovskii solution concept, wherein

$$G(t, x) := \bigcap_{\delta>0} \overline{\text{co}}[g((t, x) + \delta\overline{B}_{n+1})], \tag{3}$$

(where $\overline{\text{co}}$ denotes closed convex hull) or via the Filippov solution concept, wherein

$$G(t, x) := \bigcap_{\delta>0} \bigcap_{\text{meas}(\mathcal{N})=0} \overline{\text{co}}[g((t, x) + \delta\overline{B}_{n+1} \backslash \mathcal{N})], \tag{4}$$

the second intersection being taken over all subsets $\mathcal{N}$ of $\mathbb{R}^{n+1}$ with Lebesgue measure zero. If f is continuous and the feedback k is merely assumed to be bounded on bounded sets and (for the case of Filippov solutions) also measurable, then the multifunction G in both (3) and (4) is compact convex valued and is upper semicontinuous. Therefore one has global existence of solutions of $\dot{x} \in G(t, x)$ for any initial data. (For an overview of Krasovskii and Filippov solutions, see Filippov [11], Hajek [12], and Deimling [10]). Unfortunately, the Krasovskii and Filippov solution approaches are inadequate for purposes of (discontinuous) feedback design in stabilizability. This is due to a result of Ryan [16] (see also Clarke, Stern and Ledyaev [5]) implying that Brockett's covering condition (2) persists for these solution concepts. Specifically, in the nonholonomic integrator of Example 1.1, if a stabilizing feedback existed with respect to either of these notions of solution, then necessarily for any given $\gamma > 0$ there would have to exist $\Delta > 0$ such that

$$\Delta B_3 \subseteq G(\gamma B_3, \overline{B}_2). \tag{5}$$

But as is readily checked, this fails for the example in question, for G given by either (3) or (4).

The question of whether asymptotic controllability implies asymptotic stabilizability via feedback in some meaningful way has received some attention in recent years. One reference is Coron [9] (see also Coron and Rosier [15]). Another, which is more relevant to the present work, is Clarke, Ledyaev, Sontag and Subbotin [4], where this question was answered affirmatively, using discontinuous feedback, and a *discretized* solution concept, wherein the control function is iteratively reset and held constant on successive time intervals. The discretized solution concept utilized in the present work, that of "Euler polygonal arcs" is somewhat akin to this. The departure point of the analysis in [4] is the key fact, due to Sontag [17] (see also Sontag and Sussmann [19]) that asymptotic controllability is equivalent to the existence of a continuous (but nonsmooth) control Lyapunov function, or CLF. The methods of nonsmooth analysis were then brought to bear; the stabilizing feedback is constructed using the sublevel sets of the Moreau-Yosida infimal convolution of the CLF and exploiting a nonsmooth infinitesimal decrease property of this function. The methods of [7], instead of the sublevel sets of a CLF, the sublevel sets of the value function of the problem were utilized. A difference between the methods of [4](or [7]) and the present work is that here the Moreau-Yosida infimal convolutions is not required, and the proximal aiming method is applied. A central object in our methods is the *value function* V of the underlying problem. The value function will be discussed in some detail subsequently, but let us define it here and discuss it informally. Given an initial phase (τ, α), where $\tau < T$ and $\alpha \in \mathbb{R}^n$, $V(\tau, \alpha)$ denotes $\ell(\tilde{x}(T))$, where $\tilde{x}$ is an optimal trajectory satisfying $x(\tau) = \alpha$. By an optimal tajectory we mean a trajectory on $[\tau, T]$ corresponding to an optimal open loop control $u(t)$ (not a feedback). The plan of this article is as follows. In the next section, we provide the required preliminaries in nonsmooth analysis. Then in §3 We will show that the main theorem in [14] generalize in a meaningful way if the standing hypotheses (SH) in that result are relaxed to ;

(SH*)

(a) For each point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $F(t, x)$ is a bounded subset of $\mathbb{R}^n$.

(b) *Linear growth:* There exist $\gamma_1 > 0$ and γ_2 such that

$$\|v\| \leq \gamma_1 \|x\| + \gamma_2 \quad \forall\, v \in F(t, x), \quad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

(c) F is *upper semicontinuous* on $\mathbb{R} \times \mathbb{R}^n$; that is, given $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(t, x) - (t', x')\| < \delta \Longrightarrow F(t', x') \subseteq F(t, x) + \varepsilon B_{n+1}.$$

2. Preliminaries

2.1. **Nonsmooth analysis background.** A general reference for this section is Clarke, Ledyaev, Stern and Wolenski [8]; see also [6], Clarke [2], [3] and Loewen [13]. Let S be a nonempty subset of $\mathbb{R}^n$. The distance of a point u to S is given by

$$d_S(u) := \inf\{\|u - x\| : x \in S\}.$$

The *metric projection* of u on S is denoted

$$\operatorname{proj}_S(u) := \{x \in S : \|u - x\| = d_S(u)\}.$$

If $u \notin S$ and $x \in \operatorname{proj}_S(u)$, then the vector $u - x$ is called a *perpendicular* to S at x. The cone consisting of all nonnegative multiples of these perpendiculars is denoted $N_S^P(x)$, and is referred to as the *proximal normal cone* (or P-normal cone) to S at x.

Let $f : U \to \mathbb{R}$ be continuous, where $U \subseteq \mathbb{R}^n$ is open. Denote the epigraph of f by

$$\operatorname{epi}(f) := \{(x, y) \in U \times \mathbb{R} : x \in U,\ y \geq f(x)\}.$$

A vector $\zeta \in \mathbb{R}^n$ is said to be a *proximal subgradient* (or P-subgradient) of f at $x \in U$ provided that

$$(\zeta, -1) \in N^P_{\operatorname{epi}(f)}(x, f(x)).$$

The set of all such vectors is called the P-*subdifferential* of f at x, denoted $\partial_P f(x)$. The *limiting normal cone* (or L-normal cone) to S at $x \in S$ is defined to be the set

$$N_S^L(x) := \{\zeta : \zeta_i \to \zeta,\ \zeta_i \in N_S^P(x_i),\ x_i \to x\}.$$

We now summarize some required facts from nonsmooth calculus:

(a) *Sum rule:* Suppose that g is C^2 near a point $x \in U$. Then

$$\partial_P(g + f)(x) \subseteq g'(x) + \partial_P f(x), \tag{6}$$

where g' denotes the Fréchet derivative.

(b) *Local Lipschitzness:* Assume U to be convex as well as open. Then f is Lipschitz of rank K on U iff

$$\|\zeta\| \leq K \quad \forall\, \zeta \in \partial_P f(x) \quad \forall\, x \in U.$$

(c) *Sublevel sets:* Let f be Lipschitz on U (open and convex) and let $a \in \mathbb{R}$. Denote

$$S(a) := \{x \in U : f(x) \leq a\}.$$

Consider the differential inclusion (or generalized control system)

$$\dot{x} \in F(t, x), \tag{7}$$

where by a *solution* or *trajectory* of (7) on an interval J we mean an absolutely continuous function $t \to x(t) \in \mathbb{R}^n$ satisfying (7) a.e. on J.

3. Universal Feedback construction in optimal control

In this work we extended the main result in [14] to a situation wherein the hypotheses on that dynamics were significantly relaxed; in particular, we dropped the assumptions of convexity and closeness of the multifunction $F(t, x)$. It was also shown that the lipshitz hypotheses on $F(t, x)$ can be relaxed to upper semicontinuty.

We now define a new multifunction

$$\widehat{F}(t, x) := \overline{\text{co}}[F(t, x)].$$

One can use Carathéodory's theorem in order to show that $\widehat{F}$, which is obviously compact convex valued, is also upper semicontinuous. Consider the parametrized family of optimal control problems $\{\widehat{P}(\tau, \alpha)\}$, where $(\tau, \alpha) \in (-\infty, T] \times \mathbb{R}^n$, involving the minimization of $\ell(x(T))$ over all trajectories x of the differential inclusion $\dot{x}(t) \in \widehat{F}(t, x(t))$ satisfying $x(\tau) = \alpha$. Since compactness of trajectories holds for these dynamics, the minimum in each problem $\widehat{P}(\tau, \alpha)$ is attained, and we denote the associated value function by $\widehat{V}(\tau, \alpha)$. Also, if $\hat{f}$ is any feedback (i.e. selection) of $\widehat{F}$, then for any compact time interval $[\tau, T]$ and any $\alpha \in \mathbb{R}^n$, there exists at least one Euler solution of the initial value problem $\dot{x} = \hat{f}(x)$ satisfying $x(\tau) = \alpha$, and it is necessarily a solution of the differential inclusion $\dot{x} \in \widehat{F}(x)$. (These facts can be found in [8].)

The generalization of Theorem 3.1.2 in [14] that we wish to prove is the following.

Theorem 3.1. *Suppose that the multifunction F satisfies (SH*) and that the cost functional ℓ is continuous. Let $M > 0$ and $t_0 \in (-\infty, T)$ be given. Then given $\varepsilon > 0$, there exists a feedback f_ε for $\widehat{F} + \varepsilon B_{n+1}$ and a scalar $\tilde{\mu} > 0$ such that the following holds: Given any initial data*

$$(\tau, \alpha) \in [t_0, T] \times M\overline{B}_n \tag{8}$$

and any partition π of $[\tau, T]$ with $\operatorname{diam}(\pi) \leq \tilde{\mu}$, every Euler polygonal arc x_π of the initial value problem

$$\dot{x}(t) = f_\varepsilon(t, x(t)), \quad x(\tau) = \alpha \tag{9}$$

satisfies

$$\ell(x_\pi(T)) \leq \widehat{V}(\tau, \alpha) + \varepsilon. \tag{10}$$

Proof: As in the proof of theorem 3.1.2 in [14], we again assume $0 < t_0 < T$, and we work with a rectangle $C := [0, T] \times M\mathbb{R}^n$. As earlier, there exists $M_1 > 0$ such that for any feedback of $\widehat{F} + \varepsilon B_{n+1}$, any initial data $(\tau, \alpha) \in C$, and any partition π of $[\tau, T]$, the Euler polygonal arc x_π generated by on $[\tau, T]$ satisfies

$$\|(x_\pi(t))\| \leq M_1 \quad \forall t \in [\tau, T]. \tag{11}$$

We again denote $C_1 := [0, T] \times 2M_1\overline{B}_n$. Then for any $(\tau, \alpha) \in C$,

$$(t, (x_\pi(t))) \in [0, T] \times M_1\overline{B}_n \subseteq C_1 \quad \forall t \in [\tau, T]. \tag{12}$$

We recall a result on multifunction approximation (see Deimling [10]): Given $\varepsilon > 0$, there exists a compact convex valued multifunction $\widehat{F}^{\varepsilon}$, Lipschitz on C_1, such that

$$\widehat{F}(t,x) \subseteq \widehat{F}^{\varepsilon}(t,x) \subseteq \widehat{F}(t,x) + \varepsilon B_{n+1} \quad \forall\,(t,x) \in C_1. \tag{13}$$

Hence $\widehat{F}^{\varepsilon}$ satisfies the original standing hypotheses (SH),

(SH)

(a) For each point $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, $F(t,x)$ is a nonempty compact convex subset of $\mathbb{R}^n$.

(b) *Linear growth:* There exist positive numbers γ_1 and γ_2 such that

$$\|v\| \leq \gamma_1 \|x\| + \gamma_2 \quad \forall\, v \in F(t,x), \quad \forall\,(t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

(c) F is *locally Lipschitz* on $\mathbb{R} \times \mathbb{R}^n$; that is, to every bounded set $S \subseteq \mathbb{R} \times \mathbb{R}^n$ there corresponds $K > 0$ such that

$$F(t_1,x_1) \subset F(t_2,x_2) + K\|(t_1,x_1) - (t_2,x_2)\|B_n \quad \forall (t_i,x_i) \in S, i = 1,2,$$

and $\widehat{F}^{\varepsilon}$ is an upper approximation of $\widehat{F} = \overline{\text{co}}F$ on C_1. We denote by $\widehat{V}^{\varepsilon}$ the value function obtained when the dynamics are given by $\dot{x} \in \widehat{F}^{\varepsilon}(x)$. In view of the first containment in (13), it is clear that for any $\varepsilon > 0$ and $\widehat{F}^{\varepsilon}$ as in (13), one has

$$\widehat{V}^{\varepsilon}(\tau,\alpha) \leq \widehat{V}(\tau,\alpha) \quad \forall\,(\tau,\alpha) \in C. \tag{14}$$

Hence, it suffices to prove a version of the theorem in which the inequality (10) is replaced by

$$\ell(x_\pi(T)) \leq \widehat{V}^{\varepsilon}(\tau,\alpha) + \varepsilon \tag{15}$$

for some $\varepsilon > 0$. Observe that because $\widehat{V}^{\varepsilon}$ is associated with dynamics satisfying (SH), Lemma 3.2.7 in [14] holds true, with the notational change that the extended lower Hamiltonian $\bar{h}_F$ be replaced by $\bar{h}_{\widehat{F}^{\varepsilon}}$, and where the sublevel sets $S(a)$ are replaced by $\widehat{S}^{\varepsilon}(a)$, which are those of $(\widehat{V}^{\varepsilon})^{\beta}$, where

$$(\widehat{V}^{\varepsilon})^{\beta}(\tau,\alpha) := \widehat{V}^{\varepsilon}(\tau,\alpha) + \beta(T-\tau).$$

This version of the result will follow from the fact that for given $\varepsilon > 0$ and for any $a \in [a_m, a_M]$, one has

$$\bar{h}_{\widehat{F}^{\varepsilon}}(t,x,\eta) \leq -\frac{\beta}{\widehat{\kappa}^{\varepsilon}}\|\eta\| \quad \forall\,(t,x) \in \widehat{S}^{\varepsilon}(a) \cap \mathrm{int}(C_1), \quad \forall\,\eta \in N^P_{\widehat{S}^{\varepsilon}(a)}(t,x), \tag{16}$$

where $\widehat{\kappa}^{\varepsilon} = \kappa^{\varepsilon} + 1$ and κ^{ε} is a Lipschitz constant for $\widehat{V}^{\varepsilon}$ on (C_1). In the remainder of the proof the inequality (16) can be used in proving Theorem 3.1.2 in [14]. The remaining details are similar to the proof of that Theorem .$\Box$

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APPLICATION OF COLOMBEAU THEORY TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In classical theory of distributions(Schwartz-Sobolov theory), nonlinear operations on distributions such as product of distributions is not possible. As a result when we want to solve some partial differential equations such as Burgers one, we will encounter some ambiguities. Recently Colombeau proposed a new generalized function theory which can be used to remove these ambiguities. In this paper we consider a simplified model of elasticity and solve its equations in colmbeau theory. It is possible to handle other nonlinear partial differential equations in this framework.

1. Introduction

Classical theory of distributions, based on Schwartz-Sobolev theory of distributions, doesn't allow non-linear operations of distributions [1]. In Colombeau theory a mathematically consistent way of multiplying distributions is proposed. Colombeau's motivation is the inconsistency in multiplication and differentiation of distributions. Take, as it is given in the classical theory of distributions,

$$\theta^n = \theta \qquad \forall\ n = 2, 3, \dots, \tag{1}$$

where θ is the Heaviside step function. Differentiation of (1) gives,

$$n\theta^{n-1}\,\theta' = \theta'. \tag{2}$$

Taking $n = 2$ we obtain

$$2\theta\theta' = \theta'. \tag{3}$$

Multiplication by θ gives,

$$2\theta^2\theta' = \theta\theta'. \tag{4}$$

Using (2) it follows

$$\frac{2}{3}\theta' = \frac{1}{2}\theta' \tag{5}$$

which is unacceptable because of $\theta' \neq 0$. The trouble arises at the origin being the unique singular point of θ and θ'. If one accepts to consider $\theta^n \neq \theta$ for $n = 2, 3, \dots$, the inconsistency can be removed. The difference $\theta^n - \theta$, being infinitesimal, is the essence of Colombeau theory of generalized functions. Colombeau considers $\theta(t)$ as a function with "microscopic structure" at $t = 0$ making θ not to be a sharp step function (Fig.1), but having a width ϵ[2]. $\theta(t)$ can cross the normal axis at any value τ where $0 < \tau < 1$. With this picture in mind it is interesting to note that the behaviour of $\theta(t)^n$ around $t = 0$ is not the same as $\theta(t)$, i.e., $\theta^n(t) \neq \theta(t)$ around $t = 0$. In the following we give a short formulation of Colombeau's theory.

2. A Short Review of Colombeau Theory

Suppose $\Phi \in D(\mathbb{R}^n)$ with $D(\mathbb{R}^n)$ the space of smooth(i.e. $\mathbb{C}^\infty$) $\mathbb{C}$-valued test functions on $\mathbb{R}^n$ with compact support and

$$\int \Phi(x)dx = 1. \tag{6}$$

For $\epsilon > 0$ we define the rescaled function $\Phi^\epsilon(x)$ as

$$\Phi^\epsilon(x) = \frac{1}{\epsilon^n}\Phi(\frac{x}{\epsilon}). \tag{7}$$

Now, for $f : \mathbb{R}^n \longrightarrow \mathbb{C}$, not necessarily continuous, we define the smoothing process for f as one of the convolutions

$$\tilde{f}(x) := \int f(y)\Phi(y-x)d^n y, \tag{8}$$

or

$$\tilde{f}_\epsilon(x) := \int f(y)\Phi^\epsilon(y-x)d^n y. \tag{9}$$

According to (7), equation (9) has the following explicit form

$$\tilde{f}_\epsilon(x) := \int f(y)\frac{1}{\epsilon^n}\Phi(\frac{y-x}{\epsilon})d^n y. \tag{10}$$

This smoothing procedure is valid for distributions too. Take the distribution R, then by smoothing of R we mean one of the two convolutions (8) or (9) with f replaced by R. Remember that R is a $\mathbb{C}$-valued functional such that

$$\Phi \in D(\mathbb{R}^n) \Longrightarrow (R, \Phi) \in \mathbb{C}, \tag{11}$$

where (R, Φ) is the convolution of R and Φ.

Now we can perform the product Rf of the distribution R with the discontinuous function f through the action of the product on a test function Ψ. First we define the product of corresponding smoothed quantities $\tilde{R}_\epsilon$ with $\tilde{f}_\epsilon$ and then take the limit

$$(Rf, \Psi) = \lim_{\epsilon\to 0}\int \tilde{R}_\epsilon(x)\tilde{f}_\epsilon(x)\Psi(x)d^n x. \tag{12}$$

The multiplication so defined does not coincide with the ordinary multiplication even for continuous functions. Colombeau's strategy to resolve this difficulty is as follows. Consider one-parameter families (f_ϵ) of $\mathbb{C}^\infty$ functions used to construct the algebra

$$\begin{gathered}\mathcal{E}_{\mathcal{M}}(\mathbb{R}^n) = \{(f_\epsilon) \mid f_\epsilon \in \mathbb{C}^\infty(\mathbb{R}^n) \quad \forall K \subset \mathbb{R}^n \ \textit{compact}, \\ \forall \alpha \in \mathbb{N}^n \quad \exists N \in \mathbb{N},\ \exists \eta > 0,\ \exists c > 0 \\ \textit{such that} \ \sup_{x \in K} |D^\alpha f_\epsilon(x)| \le c\epsilon^{-N} \quad \forall 0 < \epsilon < \eta\},\end{gathered} \tag{13}$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^n)^{\alpha_n}}, \tag{14}$$

and

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Accordingly, $\mathbb{C}^\infty$-functions are embedded into $\mathcal{E}_{\mathcal{M}}$ $(\mathbb{R}^n)$ as constant sequences. For continuous functions and distributions we require a smoothing kernel $\phi(x)$, such that

$$\int d^n x \ \varphi(x) \ dx = 1 \quad \textit{and} \quad \int d^n x \ x^\alpha \varphi(x) = 0 \quad |\alpha| \ge 1. \tag{15}$$

Smoothing is defined as (10) for any function f. Now, we have to identify different embeddings of $\mathbb{C}^\infty$ functions. Take a suitable ideal $\mathcal{N}(\mathbb{R}^n)$ defined as

$$\begin{gathered}\mathcal{N}(\mathbb{R}^n) = \{(f_\epsilon) \mid (f_\epsilon) \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^n) \quad \forall K \subset \mathbb{R}^n \ \textit{compact}, \\ \forall \alpha \in \mathbb{N}^n,\ \forall N \in \mathbb{N} \quad \exists \ \eta > \mathit{0},\ \exists \ c > \mathit{0}, \\ \textit{such that} \ \sup_{x \in K} |D^\alpha f_\epsilon(x)| \le c \in^N \quad \forall 0 < \epsilon < \eta\},\end{gathered} \tag{16}$$

containing negligible functions such as

$$f(x) - \int d^n y \frac{1}{\epsilon^n} \varphi(\frac{y-x}{\epsilon}) f(y). \tag{17}$$

Now, the Colombeau algebra $\mathcal{G}(\mathbb{R}^n)$ is defined as,

$$\mathcal{G}(\mathbb{R}^n) = \frac{\mathcal{E}_{\mathcal{M}}(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)} \tag{18}$$

A Colombeau generalized function is thus a moderate family $(f_\epsilon(x))$ of $\mathbb{C}^\infty$ functions modulo negligible families. Two Colombeau objects (f_ϵ) and (g_ϵ) are said to be associate (written as $(g_\epsilon) \approx (f_\epsilon)$) if

$$\lim_{\epsilon \to 0} \int d^n x \ (f_\epsilon(x) - g_\epsilon(x)) \ \varphi(x) = 0$$

$$\forall \ \varphi \in D(\mathbb{R}^n). \tag{19}$$

For example, if $\varphi(x) = \varphi(-x)$ then $\delta\theta \approx \frac{1}{2}\delta$, where δ is Dirac delta function and θ is Heaviside Step function. Moreover, we have in this algebra $\theta^n \approx \theta$ and not $\theta^n = \theta$. For an extensive introduction to Colombeau theory, see[2-7].

3. An Example: A Simplified Model of Elasticity

In the system of elasticity Hooke's law in terms of the stress σ can be expressed as $\frac{d}{dt}\sigma = k^2 u_x$ where $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ and lower-case indices show derivatives with respect to these indices. Now the equations of system of elasticity are,

$$\rho_t + (\rho u)_x \approx 0 \quad balance\ of\ mass$$

$$(\rho u)_t + (\rho u^2)_x \approx \sigma_x \quad balance\ of\ momentum$$

$$\sigma_t + u\sigma_x \approx k^2 u_x \quad Hooke's\ law \tag{20}$$

where $\rho = density$, $u = velocity$ and $k^2 = constant$. Equations (20) are stated with three associations since we know this statement is a faithful generalization of the concept of weak solutions of systems in conservative form.

3.1. **Jump conditions.** We seek travelling waves solutions of (20) of the form

$$u(x,t) = \Delta u H(x-ct) + u_l$$

$$\sigma(x,t) = \Delta\sigma K(x-ct) + \sigma_l$$

$$\rho(x,t) = \Delta\rho L(x-ct) + \rho_l \tag{21}$$

with H, K and L three Heaviside generalized functions. Putting (21) into the first equation of (20) we get (assuming $\Delta\rho \neq 0$)

$$c - u_l = \Delta u + \rho_l \frac{\Delta u}{\Delta\rho}. \tag{22}$$

The second equation of (20) gives

$$(c - u_l - \Delta u)(u_l\Delta\rho + \rho_l\Delta u + \Delta\rho\Delta u) = u_l\rho_l\Delta u - \Delta\sigma. \tag{23}$$

These two equation are exactly the classical Rankine-Hugoniot jump conditions, since these equations are in conservative form. The last equation of (20) gives

$$c - u_l = A\Delta u - k^2\frac{\Delta u}{\Delta\sigma} \tag{24}$$

and

$$HK' \approx A\delta \tag{25}$$

where A is a real number. Now equations (22),(23) and (24) can be rewritten as

$$c = u_l + A\Delta u - k^2\frac{\Delta u}{\Delta\sigma}$$

$$k^2\frac{\Delta u}{\Delta\sigma} - \frac{1}{2}[\frac{1}{\rho_l} + \frac{1}{\rho_r}]\frac{\Delta u}{\Delta\sigma} = [A - \frac{1}{2}]\Delta u$$

$$\rho_l\rho_r(\Delta u)^2 = -\Delta\sigma\Delta\rho. \tag{26}$$

As usual we find that the jump conditions of (20) depend on an arbitrary parameter, the real number A.

3.2. **Resolution of the Ambiguities.** Now according to Colombeau theory we can state equations (20) in more precise form

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2)_x = \sigma_x \\ &\sigma_t + u\sigma_x \approx k^2 u_x. \end{aligned} \tag{27}$$

The first two equations of (27) are equivalent to

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &u_t + uu_x = \frac{1}{\rho}\sigma_x \end{aligned} \tag{28}$$

since $\rho \neq 0$. It is convenient to set $v = \frac{1}{\rho}$, where v is called the specific volume. Then (28) takes the form

$$\begin{aligned} &v_t + uv_x - vu_x = 0 \\ &u_t + uu_x - v\sigma_x = 0. \end{aligned} \tag{29}$$

Now we can restate (21) in the following form

$$\begin{aligned} u(x,t) &= \Delta u H(x - ct) + u_l \\ \sigma(x,t) &= \Delta\sigma K(x - ct) + \sigma_l \\ v(x,t) &= \Delta v M(x - ct) + v_l \end{aligned} \tag{30}$$

with $H, K, M \approx \theta$, the Heaviside function. The first equation in (29) gives

$$(c - u_l - \Delta u H)M' + \Delta u H' M + \Delta u \frac{v_l}{\Delta v} H' = 0. \tag{31}$$

The jump condition of the first equation of (27) is

$$\frac{c-u_l}{\Delta u}=\frac{\rho_r}{\Delta\rho} \qquad with \;\; \rho_r=\Delta\rho+\rho_l \tag{32}$$

which can be written as

$$\frac{c-u_l}{\Delta u}=-\frac{v_l}{\Delta v}. \tag{33}$$

Then (31) gives

$$[\frac{v_l}{\Delta v}+H]M'=[\frac{v_l}{\Delta v}+M]H'. \tag{34}$$

Putting(30) into the second equation of (29) we can obtain:

$$\frac{c-u_l}{\Delta u}H'-HH'+(\Delta vM+v_l)\frac{\Delta\sigma}{\Delta u^2}K'=0. \tag{35}$$

The jump condition of the second equation in(27) gives

$$\Delta v=\frac{(\Delta u)^2}{\Delta\sigma}. \tag{36}$$

Now we can consider equations(33),(35) and (36) together to find

$$[\frac{v_l}{\Delta v}+H]H'=[\frac{v_l}{\Delta v}+M]K'. \tag{37}$$

Setting $\alpha=\frac{v_l}{\Delta v}>0$ then (34) and (37) are the system

$$(\alpha+H)M'=(\alpha+M)H'$$

$$(\alpha+H)H'=(\alpha+M)K'. \tag{38}$$

Now these equation can be rewritten as

$$(\alpha+H)M'-H'H-\alpha H'=0$$

$$K'=\frac{\alpha+H}{\alpha+M}H'. \tag{39}$$

Since H and M are null on $(-\infty, 0[$ and identical to 1 on $]0, +\infty)$, an application of the classical formula for the solution of ordinary differential equation

$$a(x)y' + b(x)y + c(x) = 0 \tag{40}$$

allows to compute M as a function of H from the first equation of (39). One find $M = H$. This method relies upon the extension to $\mathcal{G}$ of the classical study of ordinary differential equations of the above kind. One can check that in this case the classical formula makes sense and provides a unique solution in the sense of equality in $\mathcal{G}$.

This approach can be considered as a particular case of a much deeper study of linear hyperbolic systems with coefficients in $\mathcal{G}$. There one can proves the uniqueness of the solutions of the Cauchy problem; this argument of uniqueness gives at once the result that $M = H$. THen the second equation in (39) gives $K = H$. Now we have resolved the ambiguities,

$$HK' = HH' = \frac{1}{2}\delta \tag{41}$$

and therefore $A = \frac{1}{2}$.

As a conclusion to above argument we can state the following theorem:

Theorem

The system of two equations and three unknowns

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2)_x - \sigma_x = 0 \tag{42}$$

is equivalent to the system$(v = \frac{1}{\rho})$

$$v_t + uv_x - vu_x = 0$$

$$u_t + uu_x - v\sigma_x = 0. \tag{43}$$

Further, travelling waves of the form

$$w(x,t) = \Delta w H_w(x - ct) + w_l \tag{44}$$

with $w = v, u, \sigma$ successively (Δw, c, $w_l \in \mathbb{R}$ and H_w a Heaviside generalized function), are solution of (43) if and only if $H_v = H_u = H_\sigma$ plus the classical jump condition of (42).

4. Conclusions

Since the classical theory of distributions is disable to handling nonlinear operations on distributions, Colombeau theory of generalized functions give a reasonable framework to do such nonlinear operations and as a result this theory can be used to remove ambiguities of classical theory. In this paper we have used this new theory to solve the equations of a system of elasticity. Although physically this problem must have unambiguous traveling waves solutions, classical theory of distributions for such a solutions encounter some ambiguities. In the line of removing these ambiguities we arrived at the important result between Heaviside step function and Dirac δ-function, relation (41).

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CONVOLUTION ON HOMOGENEOUS SPACES

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Abstract. We will show that the notion of convolution can be extented, in a natural way, to homogeneous spaces. with only minor constraints.This convolution maintains the basic properties of the classical convolution on locally compact topological groups. This construction paves the turn can be used in many applications.

1. Preliminaries

Let X be a (locally compact) topological space. and let S be a semigroup acting on X, i.e. ther exits a mapping (called action) $(s, x) \longrightarrow sx : S \times X \longrightarrow X$ such that $s(tx) = (st)x$ for all $s, t \in S,\ x \in X$. S is said to act transitively on X if for each pair $x, y \in X$, there exists an element $s \in S$, such that $sx = y$. As a special case we shall consider homogeneous spaces, i.e. the case wher a locally compact topological group G, acts transitively on a locally compact topological space X, such that the action is jointly continuous, and for every $x \in X$ the mapping $\pi_x : g \longrightarrow gx : G \longrightarrow X$ is an open mapping of$G$onto X. if so, then X is homeomorphic to G/Gx, for every $x \in X$, where $G_x = \{g \in G : gx = x\}$ is a subgroup of G. For this reason, in the sequel we shall denote any of the mappings $g \longrightarrow gx$ simply by π, without referring to π. A positive regular Borel measure μ on X is called G-invariant if for each Borel subset B of X, and each $g \in G$, we have

$$\mu(gB) = \mu(B)$$

where $gB = \{gx : x \in B\}$. μ is called relatively G-invariant if ther exists a continuous mapping $\Delta : G \longrightarrow \mathbb{R}^+$, such that,

$$\mu(gB) = \Delta(g)\mu(B).$$

It can be proved that Δ is homomorphism into $\mathbb{R}^+$, the multiplicative group of positive real numbers. It can also be proved that

$$\int_X f(x)d\mu(x) = \Delta(g)\int_X f(gx)d\mu(x).$$

The existence of a G-invariant or a relatively G-invariant measure depends both on G, and X, and also on the nature of the action. Let X be a homogeneous space relative to the topological group G, and let the mapping $\pi : G \longrightarrow X$ be a proper one, i.e. the image of each compact subset of G under π is compact in X. Then for each $f \in C_{00}(X)$, the space of functions of compact support on X, we have $f \circ \pi \in C_{00}(G)$. Suppose also that X carries a G-invariant measure μ. By choosing asuitable factor of proportionality, we can have suitable Haar measure dg, on G such that for each $f \in C_{00}(X)$

$$\int_G f \circ \pi(g)dg = \int_X f(x)d\mu(X).$$

under the above assumptions we can make the following definition. Notice that the condition of unimodularity is added for convenience only, and can be avoided by introducing some modifications.
The background material we will need can be found in [2],[3], and [4].

2. Definition. Let X and G be as above. Furthermore, suppose that G is a unimodular group. Let $1 \leq p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then for

$f \in L^p(G), h \in L^q(X)$, we define the convolution of f and h by

$$f * h(x) = \int_G f(g)h(g^{-1}x)dg \quad (x \in X).$$

Using Holder's inequality, we can prove that the above integral exists, and in fact is bounded by $||f||_p||h||_q$. Furthermore, we have

3. Theorem. *if f and g are as above, and $1 < p < \infty$, then $f * h \in C_0(X)$.*

Proof. We first prove that $f*h$ is continuous. Let $x_0 \in X$ be an arbitary point and let $h_{x_0}(g) = h(gx_0)$ for $g \in G$. Then supposing that $x = g_0x_0$, once again by Holder's ineqality

$$|f * h(x) - f * h(x_0)| = |\int_G [f(g)h(g^{-1}x) - f(g)h(g^{-1}x_0)]dg|$$

$$\leq ||f||_p(\int_G |h(g^{-1}x) - h(g^{-1}x_0)|^q dg)^{\frac{1}{q}}$$

$$= ||f||_p(\int_G |h_{x_0}(g^{-1}g_0) - h_{x_0}(g^{-1})|^q)^{\frac{1}{q}}$$

$$= ||f||_p(\int_G |(h_{x_0})g_0(g^{-1}) - h_{x_0}(g^{-1})|^q)^{\frac{1}{q}}$$

$$= ||f||_p||(h_{x_0})g_0 - h_{x_0}||_q$$

wich tends to zero as x tends to x_0 (i.e. as g_0 tends to e, the identity element of G). notice that we used the unimodularity of G fir obtaining the last equality. However without this aasumption it is not hard to prove the continuity of $f * h$. Now to prove $f * h$ vanishes at infinity. Let $\varepsilon > 0$ be arbitrary, and $K_1 \subset G$, and $K_2 \subset X$ be compact sets such that

$$\int_{G\setminus K_1} |f(g)|^p dg < \epsilon^p \quad \textit{and} \quad \int_{X\setminus K_2} |h(x)|^q d\mu(x) < \epsilon^p.$$

Obviously if $x \notin K_1K_2$, then $g^{-1}x \notin K_2$ for any $g \in K_1$. Therefore

$$|f * h(x)| \leq \int_G |f(g)h(g^{-1}x)|dg$$

$$\leq \int_{K_1} |f(g)h(g^{-1}x)|dg + \int_{G\backslash K_1} |f(g)h(g^{-1}x)|dg$$

Once again by Holder's inequality

$\int_{G\backslash K_1} |f(g)h(g^{-1}x)|dg \leq (\int_{G\backslash K_1} |f(g)|^p dg)^{\frac{1}{p}} (\int_{G\backslash K_1} |h(g^{-1}x)|^q dg)\frac{1}{q}$

$$\varepsilon||h||_q$$

We also have

$$\int_{K_1} |f(g)h(g^{-1}x)|dg \leq (\int_{K_1} |f(g)|^p dg)^{\frac{1}{p}} (\int_{K_1} |h(g^{-1}x)|^q dg)\frac{1}{q}$$

$$\| f||_p(\int_{G\backslash K_2} |h(y)|^q d\mu(y))^{\frac{1}{q}}$$

$$\varepsilon \parallel f \parallel_p .$$

So for $x \notin K_1K_2, |f * h(x)| < \varepsilon(\parallel f \parallel_p + \parallel h \parallel_q)$. since K_1K_2 is a compact subset of X, this means tath $\lim_{x\to\infty} f * h(x) = 0$. We now try to define the notion of a Fourier algebra, for a homogeneous space. This extends the notion of Fourier algebra of a (non-abelian) locally compact topological group as defined by Eymard[1]. Let G and X be as above, and let $T(G, X)$ denote the projective tensor product $L^2(G \otimes L^2(X)$, and denot the norm of an element $\varphi \in T(G, X)$ by $\parallel \varphi||_T$. By definition of projective tensor product, given any $\varepsilon > 0$, and $\varphi \in T(G, X)$, there exist a sequence $a_n \subset C$, and tow sequences $f_n \subset L^2(G)$, and $h_n \subset L^2(X)$ such that

$$\varphi(g, x) = \sum_{n=1}^{\infty} a_n f_n(g) h_n(x) \quad (g \in G, x \in X)$$

and

$$\sum_{n=1}^{\infty} |a_n|||f_n \parallel_2 ||h_n \parallel_2 \leq ||\varphi||_T + \varepsilon.$$

Define the mapping $P : T(X, G) \longrightarrow C(X)$, by

$$P(\varphi)(x) = \int_G \varphi(g, g^{-1}x)dg$$

In other words,P is the mapping induced on$T(G,X)$ by

$$P(f*h)(x)=\int_G f(g)h(g^{-1}x)dg$$

i.e. $P(f\otimes h)=f*g$. So as we have already seen $||P(f\otimes h||_infty \leq ||f\,\|_2\,||h||_2$, which implies $\|\,P||=1$.

4. Definition. We denote by $A(G,X)$, the Banach space $T(G,X)/P^{-1}(0)$, and we call it the Fourier algebra of X, with respect to G. The norm of element $f\in A(G,X)$ is denoted by $\|\,f\,\|_A$. It should be noticed that although we can think of elements of $A(G,X)$ as functions on $G\times X$, but the induced isomorphism $A(G,X)\longrightarrow C(X)$ allows us to corsider it as a subspace of $C(X)$, but normed with the quotient norm $\|\quad||_A$. Obvously for $\varphi\in T(G,X)$, we have

$$\|\,P_\varphi\,\|\leq ||\varphi||_T.$$

5. Theorem. $A(G,X)\subset C_0(X)$*, and for* $\varphi\in A(G,X)$*, we have*

$$\|\,\varphi||_\infty\leq ||\varphi||_A.$$

Proof. By Theorem 3, for $f\in L^2(G), h\in L^2(X), P(f\otimes h)=f*h\in C_0(X)$. therefore

$$P(L^2(G)\otimes L^2(X))\subset C_0(X).$$

Since P is continuous, and $C_0(X)$ is a complete linear space, we can extend P to the completion of $L^2(G)\otimes L^2(X)$, i.e. $L^2(G)\otimes L^2(X)$ and the image of P remains inside $C_0(X)$. The inequality $\|\,\varphi||_\infty\leq ||\varphi||_A$. is a consequence of the inequality $\|\;P(f\otimes h)\;\|\leq\;||f||_2\;\|\;h||_2$, and the definition of the quotient norm. The space $A(G,X)$ has many nice propeties, among which we mention that it is a Banach algebra. We hope to use these properties in wavelet theory, and signal processing an upcoming article.

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ON THE DERIVATIVE OF HERMITIAN DISTANCES

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ABSTRACT. We will consider the " derivative " of the distance induced by a Hermitian metric on complex manifolds. This notion will be applied to some function spaces on complex manifolds and a theorem in classical function theory will be generalized.

1. INTRODUCTION

Let M be a connected complex manifold and TM be its complex tangent bundle. A differential metric on M is a function $f : M \longrightarrow \mathbb{R}_{\geq 0}$ satisfying the following condition:

$$f(aX_x) = |a|f(X_x), \text{ for each } X_x \in T_xM \text{ and } a \in \mathbb{C}.$$

An upper semicontinuous differential metric is called a Finsler metric.

If we consider the integrated form of a Finsler metric f, i. e. the function $F : M \times M \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$$F(x, y) = \inf\{\int_0^1 f(\gamma'(t))dt\},$$

where the infimum is taken with respect to set of piecewise differentiable curves $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$, then F is a pseudo-distance.

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Note that, if h is a Hermitian metric on a complex manifold M, then the function $\tilde{h} : TM \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\tilde{h}(X_x) := h(X_x, X_x)^{1/2}, \qquad \forall X_x \in T_x M,$$

is a continuous Finsler metric.

The second example is given by the Kobayashi-Royden [5] metric k. This Finsler metric is defined by

$$k(X_x) := \inf\{a > 0 : \exists \varphi : \mathbb{D} \longrightarrow M, \varphi(0) = x \text{ and } \varphi'(a(\frac{\partial}{\partial z})_0) = X_x\}.$$

A complex manifold M is called hyperbolic if the pseudo-distance K induced by the differential metric k is a distance.

The derivative of a pseudo-distance d on a complex manifold M is a function $F(d)$ defined on the tangent bundle TM by

$$F(d)(X_x) := \limsup_{t \to 0} \frac{d(\gamma(t), \gamma(0))}{|t|}, \tag{1}$$

where γ is a curve in M defined in a neighborhood of 0 in $\mathbb{R}$ with $\gamma(0) = x$ and $\gamma'(0) = X_x$.

One can prove that (1) does not depend on the curve γ and $F(d)$ is a differential metric on M. We will prove that the derivative of a Hermitian distance on a complex manifold M, induced by a Hermitian metric is in fact equal to that metric.

Let M and N are connected complex manifolds of dimensions m and n with differential metrics F_M and F_N, respectively.

We say that a mapping $f \in \mathcal{H}(M, N)$ is of bounded expansion if it satisfies

$$\|f'\| := \sup\{\|f'(p)\| : p \in M\} < \infty, \tag{2}$$

where

$$\|f'(p)\| := \sup\{F_N(f'(p)X_p) : X_p \in T_p M,\ F_M(X_p) = 1\}. \tag{3}$$

Usually, M will be hyperbolic and F_M will be the Kobayashi-Royden metric k_M while N will be Hermitian and F_N will be the Hermitian metric h_N. The class of mappings of bounded expansion will be denoted by $\mathcal{E}(M,N)$.

Let F_1 and F_2 be two differential metric such that for each $0_p \neq X_p \in T_pM$

$$F_i(X_p) \neq 0 \qquad i = 1, 2.$$

Then

$$\begin{aligned} \sup_{X_p \neq 0_p} \frac{h_N(f'(p)X_p)}{F_1(X_p)} &= \sup_{F_2(X_p)=1} \frac{h_N(f'(p)X_p)}{F_1(X_p)} \\ &= \sup_{F_1(X_p)=1} h_N(f'(p)X_p). \end{aligned} \tag{4}$$

If M is hyperbolic and (p,ξ) represents the tangent vector $X_p \in T_pM$ in a coordinate neighborhood of $p \in M$ then, $k_M(p,\xi) \neq 0$ for $\xi \neq 0$, and by (4) we have

$$\|f'(p)\| = \sup_{k_M(p,\xi)\neq 0} \frac{h_N(f(p), f'(p)\xi)}{k_M(p,\xi)},$$

$$= \sup_{|\xi|=1} \frac{h_N(f(p), f'(p)\xi)}{k_M(p,\xi)}, \tag{5}$$

where $|\quad|$ is the Euclidean norm in $\mathbb{C}^m$.

Note that (5) does not depend on the coordinate neighborhood around $p \in M$.

If N is noncompact, we refer to mappings of bounded expansion as Bloch mappings and if N is compact we refer to them as normal mappings.

For the case of normal mappings our definition is consistent with Hahn's definition of normal mappings in [2]. One can verify that our definitions coincide with the classical definitions of Bloch and normal functions, respectively.

2. HERMITIAN DISTANCES

Theorem 2.1. *Let Ω be a domain in $\mathbb{C}^n$, $\delta : \Omega \times \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}$ a continuous function such that for each $a \in \Omega$, $\delta(a, .)$ is a $\mathbb{C}$-norm and let d be its corresponding distance. Then, for each $a \in \Omega$ and $\xi \in \mathbb{C}^n$,*

$$\lim_{t \to 0^+} \frac{1}{t} d(a, a + t\xi) = \delta(a, \xi). \tag{6}$$

Proof. By definition $d(a, b) := \inf\{\int_0^1 \delta(\alpha(\tau), \alpha'(\tau))d\tau\}$ where the infimum is taken over all piecewise smooth curves $\alpha : [0, 1] \longrightarrow \Omega$ joining a and b. Fix $a \in \Omega$ and $\xi \in \mathbb{C}^n$, $\xi \neq 0$. Taking $\alpha(\tau) := a + t\tau\xi$, we see that

$$\begin{aligned}
\limsup_{t \to 0^+} \frac{1}{t} d(a, a + t\xi) &\leq \limsup_{t \to 0^+} \frac{1}{t} \int_0^1 \delta(a + t\tau\xi, t\xi)d\tau \\
&= \limsup_{t \to 0^+} \frac{1}{t} \int_0^t \delta(a + s\xi, \xi)ds \\
&\leq \delta(a, \xi).
\end{aligned}$$

So far, we have used the fact that δ is upper semicontinuous.

Conversely, fix $0 < \theta < 1$. Since δ is continuous, for $\eta_0 \in \mathbb{C}^n$ with $|\eta_0| = 1$, there exist neighborhoods $U_{\eta_0} \subset \Omega$ of a and $V_{\eta_0} \subset \mathbb{C}^n$ of η_0 such that $\delta(z, \eta) \geq \theta\delta(a, \eta)$ for each $z \in U_{\eta_0}$ and $\eta \in V_{\eta_0}$.

Since the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{C}^n$ is compact one can find a neighborhood U of a such that $\delta(z, \eta) \geq \theta\delta(a, \eta)$ for each $z \in U$ and $\eta \in \mathbb{S}^{n-1}$. Since $\delta(z, .)$ is a $\mathbb{C}$-norm, it follows that for each $z \in U$ and $\eta \in \mathbb{C}^n$, $\delta(z, \eta) \geq \theta\delta(a, \eta)$.

For small $t > 0$, there exists a geodesic $\gamma_t : [0,1] \longrightarrow \tilde{U} \subset U$, with $\gamma_t(0) = a$, $\gamma_t(1) = a + t\xi$. Therefore

$$\begin{aligned} d(a, a+t\xi) &= \int_0^1 \delta(\gamma_t(\tau), \gamma_t'(\tau))d\tau, \\ &\geq \theta \int_0^1 \delta(a, \gamma_t'(\tau))d\tau, \\ &\geq \theta\delta(a, \int_0^1 \gamma_t'(\tau)d\tau), \\ &= \theta\delta(a, \gamma_t(1) - \gamma_t(0)), \\ &= \theta\delta(a, t\xi). \end{aligned}$$

The second inequality follows from the fact that $\delta(a, .)$ is a complex norm. It follows that

$$\liminf_{t\to 0^+} \frac{1}{t} d(a, a+t\xi) \geq \theta\delta(a, \xi).$$

Hence, the proof is complete. □

Lemma 2.2. *Let Ω and δ be as in the previous lemma, $\Omega_1 \subset \mathbb{C}^m$ and $f : \Omega_1 \longrightarrow \Omega$ a holomorphic mapping. Then, for $a \in \Omega_1$ and $\xi \in \mathbb{C}^m$,*

$$\lim_{t\to 0^+} \frac{1}{t} d(f(a), f(a+t\xi)) = \delta(f(a), f'(a)\xi).$$

Proof. We shall prove that for each $b \in \Omega$, there exist $M, r > 0$, such that $d(z', z'') \leq M\|z' - z''\|$, for all $z', z'' \in \mathbb{B}_r(b) \subset G$. To see this, fix $b \in \Omega$ and choose $r > 0$ such that $\bar{\mathbb{B}}_r(b) \subset \Omega$. Since δ is upper semicontinuous, there exists $M > 0$ such that $\delta(z, \xi) \leq M\|\xi\|$, for each $z \in \bar{\mathbb{B}}_r(b)$ and each $\xi \in \mathbb{C}^n$. It follows that for $z', z'' \in \mathbb{B}_r(b)$,

$$d(z', z'') \leq \int_0^1 \delta(z' + t(z'' - z'), z'' - z')dt \leq M\|z' - z''\|. \tag{7}$$

We claim that

$$\delta(b, \xi_0) = \lim_{\substack{t\to 0+ \\ \xi\to\xi_0}} \frac{1}{t} d(b, b+t\xi). \tag{8}$$

Choose M and r as above. Then for $0 < t < (r/2)/(\|\xi_0\| + r/2)$ and $\xi \in \mathbb{B}_{r/2}(\xi_0)$ we have $\|b + t\xi - b\| < r/2$. Hence, by (7),

$$d(b + t\xi_0, b + t\xi) \leq Mt\|\xi - \xi_0\|.$$

Therefore,

$$\begin{aligned} d(b, b + t\xi) &\leq d(b, b + t\xi_0) + d(b + t\xi_0, b + t\xi) \\ &\leq d(b, b + t\xi_0) + Mt\|\xi - \xi_0\|. \end{aligned}$$

Similarly,

$$d(b, b + t\xi_0) \leq d(b, b + t\xi_0) + Mt\|\xi - \xi_0\|. \tag{9}$$

Applying the previous theorem we have,

$$\lim_{\substack{t \to 0^+ \\ \xi \to \xi_0}} \frac{1}{t} d(b, b + t\xi) = \lim_{t \to 0^+} \frac{1}{t} d(b, b + t\xi_0) = \delta(b, \xi_0). \tag{10}$$

Hence our claim in (8) is proved. With the help of this fact, for $a \in \Omega_1$ and $\xi \in \mathbb{C}^m$ we have,

$$\begin{aligned} \lim_{t \to 0^+} \frac{1}{t} d(f(a), f(a + t\xi)) &= \lim_{t \to 0^+} \frac{1}{t} d(f(a), f(a) + t(f'(a)\xi + o(1))), \\ &= \delta(f(a), f'(a)\xi). \end{aligned}$$

□

Theorem 2.3. *Let M and N be hyperbolic and Hermitian manifolds, respectively. Then the function $f \longmapsto \|f'\|$ from the class of holomorphic bounded expansion mappings $\mathcal{E}(M, N)$ equipped with the compact open topology to $\mathbb{R}$ is lower semicontinuous.*

Proof. Let $\{f_n\}$ be a sequence in $\mathcal{E}(M, N)$ which converges to a holomorphic mapping f uniformly on compact subsets of M. Let $\{\|f'_{n_k}\|\}$

be a subsequence of $\{\|f_n'\|\}$ converging to α. For a given $\varepsilon > 0$ and $p,\ q \in M$, $p \neq q$, there exists $n_\varepsilon \in \mathbb{N}$ such that for $n_k > n_\varepsilon$ we have

$$|\ \|f_{n_k}'\| - \alpha| < \frac{\varepsilon}{2K_M(p,q)}, \tag{11}$$

and

$$d_N(f_{n_k}(p), f(p)) \leq \varepsilon/4, \qquad d_N(f_{n_k}(q), f(q)) \leq \varepsilon/4. \tag{12}$$

On the other hand, since for each $n \in \mathbb{N}$, $f_n \in \mathcal{N}(M,N)$, according to the definitions of $\|f_n'(x)\|$ and $\|f_n'\|$, for each $x \in M$ and $\xi \in \mathbb{C}^m$

$$h_N(f_n(x), f_n'(x)\xi) \leq \|f_n'\| k_M(x,\xi).$$

By integrating along any C^1 curve connecting p to q, we have

$$d_N(f_n(p), f_n(q)) \leq \|f_n'\| K_M(p,q).$$

Hence employing (11) and (12), for $n_k > n_\varepsilon$ we obtain

$$\begin{aligned} d_N(f(p), f(q)) &\leq d_N(f(p), f_{n_k}(p)) + d_N(f_{n_k}(p), f_{n_k}(q)) \\ &\quad + d_N(f_{n_k}(q), f(q)) \\ &\leq \varepsilon/2 + \|f_{n_k}'\| K_M(p,q) < \varepsilon + \alpha K_M(p,q). \end{aligned}$$

Therefore

$$d_N(f(p), f(q)) \leq \alpha K_M(p,q). \tag{13}$$

Now applying Lemma 2.2 and also the first part of Theorem 2.1 locally, for each $p \in M$ and each $\xi \in \mathbb{C}^m$, we have

$$\begin{aligned} h_N(f(p), f'(p)\xi) &= \lim_{t \to 0^+} \frac{1}{t} d_N(f(p), f(p+t\xi)) \\ &\leq \alpha \limsup_{t \to 0^+} \frac{1}{t} K_M(p, p+t\xi) \\ &\leq \alpha k_M(p,\xi). \end{aligned}$$

It follows that $f \in \mathcal{E}(M,N)$ and $\|f'\| \leq \alpha$. The assertion is thus proved.

$\square$

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ON PROJECTIVE LIMITS OF SEQUENCES OF BANACH FUNCTION ALGEBRAS

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ABSTRACT. Let X be a hemicompact space with (K_n) as an admissible exhaustion and let $(A_n, \|.\|_n)$ be a sequence of Banach function algebras such that $A_n \subseteq C(K_n)$ and $A_{n+1}|_{K_n} \subseteq A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$, $n \in \mathbb{N}$, $f \in A_{n+1}$. We define a new algebra $A = \{f \in C(X) : f|_{K_n} \in A_n, n \in \mathbb{N}\}$ and show that if A separates the points of X then it is a Fréchet function algebra on X under some topology. In the case that each A_n is natural we give a result related to the spectrum of A. We also show that if X is a hemicompact noncompact space then a closed subalgebra of A can not be normable as a regular Banach function algebra. As an application of the results the Lipschitz algebra of infinitely differentiable functions is considered.

1. Introduction

Let X be a compact Hausdorff space. We denote the algebra of all continuous functions on X by $C(X)$ and the uniform norm of $f \in C(X)$ by $\|f\|_X$. A subalgebra of $C(X)$ which contains the constants and separates the points of X and is a Banach algebra under a norm is called a *Banach function algebra* on X. The uniform norm of an element in a

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Banach function algebra does not exceed from its norm. If the norm of a Banach function algebra is equivalent to the uniform norm then it is called a *uniform (Banach) algebra.*

A topological algebra is called a *locally multiplicatively convex* algebra (LMC-algebra) if there is a local basis (U_α) of convex neighborhoods of the origin such that each U_α is submultiplicative (that is $U_\alpha.U_\alpha \subseteq U_\alpha$). A topological algebra A is called a *Q-algebra* if the set of quasi-regular elements of A is open in A. This is equivalent to say that the set of all quasi-regular elements of A has an interior [1].

By a Fréchet algebra we mean an LMC-algebra which is moreover complete and metrizable. So the topology of a Fréchet algebra can be defined by a sequence (p_n) of separating, submultiplicative seminorms on it $(p_n(fg) \leq p_n(f)p_n(g), f, g \in A)$. Without loss of generality we can assume that $p_n \leq p_{n+1}$ and $p_n(1) = 1$, if A has unit. We denote a Fréchet algebra A with this sequence of seminorms by $(A, (p_n))$.

In this paper we assume that all algebras are unital.

Let $(A, (p_n))$ be a commutative Fréchet algebra. The set of all non-zero complex homomorphisms on A is denoted by S_A and the *spectrum* of A, denoted by M_A, is the set of all non-zero continuous complex homomorphisms on A, and for every $f \in A$, $\hat{f} : S_A \longrightarrow \mathbb{C}, \psi \longmapsto \psi(f)$ is the Gelfand transform of f and $\hat{A} = \{\hat{f}|_{M_A} : f \in A\}$. We always endow S_A or M_A with the Gelfand topology. The Fréchet algebra A is called *functionally continuous* if $M_A = S_A$. As we know $M_B = S_B$ for any Banach algebra B. But it is unanswered whether or not each Fréchet algebra is functionally continuous (Michael's problem).

When B is a Banach function algebra on X we can consider X as a subset of M_B through the map $J : X \longrightarrow M_B, x \longmapsto \varphi_x$ where φ_x is the evaluation homomorphism at x. We say that B is *natural* if J is onto (and hence a homeomorphism).

Let $(A,(p_n))$ be a Fréchet algebra. It is a classical result that A can be represented as a projective limit of a sequence of Banach algebras. That is, there exists a sequence of Banach algebras $(A_n,\|.\|_n)$ and continuous algebra homomorphisms $\varphi_n : A_{n+1} \longrightarrow A_n$, $n \in \mathbb{N}$, with dense ranges such that $A = \varprojlim A_n$ topologically and algebraically where the projective limit $\varprojlim A_n = \{(f_n)_n \in \Pi A_n : \varphi_n(f_{n+1}) = f_n, n \in \mathbb{N}\}$ is endowed with the coordinatewise operations and the relative product topology. Indeed each A_n can be considered as the completion of $A/\ker p_n$ with respect to the norm $p'_n(f + \ker p_n) = p_n(f)$. Moreover $M_A = \bigcup M_{A_n}$ (as sets) or more precisely $M_A = \bigcup \pi_n^*(M_{A_n})$ where $\pi_n : A \longrightarrow A_n, a \longmapsto a + \ker p_n$ and $\pi_n^* : M_{A_n} \longrightarrow M_A$ defined by $\pi_n^*(\varphi) = \varphi \circ \pi_n$ is a homeomorphism onto its image. For more details one can refer to [3].

A Hausdorff space X is called *hemicompact* if there exists a sequence (K_n) of increasing compact subsets of X such that each compact subset of X is contained in some K_n. The sequence (K_n) with the above property is called an *admissible exhaustion* of X.

A Hausdorff space X is said to be a *k-space* if every subset intersecting each compact subset in a closed set is itself closed. So a function defined on a k-space X is continuous if and only if it is continuous on each compact subset of X.

Definition 1.1. A commutative Fréchet algebra A is called a *uniform Fréchet algebra (uF-algebra)* if there exists a generating sequence of submultiplicative seminorms (p_n) for its topology such that $p_n(f^2) = p_n(f)^2$, for all $f \in A$ and $n \in \mathbb{N}$.

Clearly each uniform (Banach) algebra is a uF-algebra.

If X is a hemicompact k-space and (K_n) an admissible exhaustion of X then $C(X)$ is a uF-algebra under the compact-open topology which is the topology generated by the sequence of the seminorms $(\|.\|_{K_n})_n$ and $C(X) = \varprojlim C(K_n)$.

If $(A, (p_n))$ is a commutative Fréchet algebra and (A_n) is a sequence of Banach algebras such that $A = \varprojlim A_n$ then M_A is a hemicompact space and (M_{A_n}) is an admissible exhaustion of M_A [3]. But there is an example which shows that in general M_A is not a k-space [3].

If $(A, (p_n))$ is a uF-algebra then by identifying A with $\hat{A}$, where $\hat{A}$ is endowed with the relative compact-open topology, we can consider A as a point separating and complete subalgebra of $C(X)$ which also contains the constants. Moreover in this case A is the projective limit of a sequence of uniform (Banach) algebras [3].

Clearly when A is a uF-algebra, each evaluation homomorphism $\varphi_x : A \longrightarrow \mathbb{C}$, $f \longmapsto f(x)$ is continuous and the structure map $J : X \longrightarrow M_A, x \longmapsto \varphi_x$ is injective and continuous.

Definition 1.2. Let X be a hemicompact space and A a subalgebra of $C(X)$ which contains the constants and separates the points of X. We call A a *Fréchet function algebra* (Ff-algebra) on X if it is a Fréchet algebra with respect to some topology such that for every $x \in X$ the evaluation homomorphism at x is continuous, that is $\varphi_x \in M_A, x \in X$.

Remark. (a) Clearly each Ff-algebra is a commutative (unital) semisimple Fréchet algebra. Conversely if $(A, (p_n))$ is a commutative (unital) semisimple Fréchet algebra then by identifying $(A, (p_n))$ with Fréchet algebra $(\hat{A}, (\hat{p}_n))$, where $\hat{p}_n(\hat{f}) = p_n(f)$, $f \in A$, we can consider A as an Ff-algebra on its spectrum. So indeed the class of Ff-algebras and the class of commutative (unital) semisimple Fréchet algebras are the same.

(b) Each uF-algebra and each Banach function algebra is an Ff-algebra.

Let $(A,(p_n))$ be an Ff-algebra on X and for each $n \in \mathbb{N}, A_n$ be the completion of $A/\ker p_n$ with respect to the norm $p_n'(f+\ker p_n) = p_n(f)$. Then $M_{A_m} = \{\varphi \in M_A : |\varphi(f)| \leq p_m(f), f \in A\}$, $m \in \mathbb{N}$ and (M_{A_m}) is an admissible exhaustion of M_A [3]. Now since $J : X \longrightarrow M_A, x \longmapsto \varphi_x$ is continuous (and injective) $\{\varphi(x) : x \in K_n\}$ is a compact subset of M_A, for each $n \in \mathbb{N}$, and so there exists some m such that $\{\varphi_x : x \in K_n\} \subseteq M_{A_m}$. Therefore

$$\|f\|_{K_n} = \sup_{x\in K_n} |\varphi_x(f)| \leq \sup_{\varphi \in M_{A_m}} |\varphi(f)| \leq \|\hat{f}\|_{M_{A_m}} \leq p_m(f) \quad (f \in A). \quad (1)$$

For $n \in \mathbb{N}$, let $i(n) \geq n$ be the smallest integer that $\|f\|_{K_n} \leq p_{i(n)}(f)$ holds for all $f \in A$ and define p_n'' on $A|_{K_n}$ by

$$p_n''(f|_{K_n}) = \inf\{p_{i(n)}(g) : g|_{K_n} = f|_{K_n}, g \in A\} \quad (f \in A).$$

Then p_n'' is an algebra norm on $A|_{K_n}$. Let A_{K_n} be the completion of $A|_{K_n}$ with respect to the norm p_n''. Then we have the following theorem:

Theorem 1.3. [6] *Let $(A,(p_n))$ be an Ff-algebra on X, and (K_n) be an admissible exhaustion of X and the sequence (A_{K_n}) be defined as above. Then (A_{K_n}) is a sequence of Banach algebras, where each A_{K_n} contains $A|_{K_n} \subseteq C(K_n)$ as a dense subalgebra and A is dense in $\varprojlim A_{K_n}$. Moreover if for each $n \in \mathbb{N}$, $\ker q_n \subseteq \ker p_{i(n)}$, where q_n is defined by $q_n(f) = \|f\|_{K_n}$, then $A = \varprojlim A_{K_n}$ (topologically and algebraically).*

Theorem 1.4. [6] *Let $(A,(p_n))$ and $(B,(q_n))$ be Ff-algebras on hemi-compact spaces X and Y, respectively, and let $T : (A,(p_n)) \longrightarrow (B,(q_n))$ be a continuous monomorphism with a dense range. Then the continuous and injective adjoint spectral map $T^* : M_B \longrightarrow M_A, \psi \longmapsto \psi \circ T$ is surjective and proper (that is the inverse image of each compact set is compact) if and only if for each m there exists some n such that*

$$\|\hat{f}\|_{M_{A_m}} \leq q_n(T(f)) \quad (f \in A).$$

2. Main results

As we mentioned before each uF-algebra is a projective limit of a sequence of uniform (Banach) algebras. There is an example which shows that in general a commutative semisimple Fréchet algebra with unit (which we called it an Ff-algebra) can not be represented as a projective limit of a sequence of Banach function algebras [5]. In the following we consider special Fréchet function algebras such that they can be represented as projective limits of sequences of Banach function algebras and obtain a result related to their spectrums.

Let X be a hemicompact space and (K_n) an admissible exhaustion of X. Let (A_n) be a sequence of Banach function algebras such that for each $n \in \mathbb{N}$, A_n is a Banach function algebra on K_n with respect to $\|.\|_n$ and $A_{n+1}|_{K_n} \subseteq A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$ for all $f \in A_{n+1}$. Now define a subalgebra A of $C(X)$ as follows:

$$A = \{f \in C(X) : f|_{K_n} \in A_n, n \in \mathbb{N}\}.$$

Clearly A contains the constants and for each $n \in \mathbb{N}$,

$$p_n(f) = \|f|_{K_n}\|_n, f \in A$$

defines a submultiplicative seminorm on A. It is straightforward to check that A is a Fréchet algebra with respect to the topology defined by the sequence (p_n) of seminorms. Moreover for each $x \in X$, φ_x, the evaluation map at x is continuous. So if A separates the points of X, then it will be an Ff-algebra on X.

Note that when X is compact and each A_n is inverse closed, that is $1/f \in A_n$ if $f \in A_n$ and $f(x) \neq 0$ for all $x \in K_n$, then we can verify easily that A is a Q-algebra. Because clearly A is also inverse closed and for some N, $K_n = X$, for $n \geq N$. If we set $G = \{f \in A : 1 + f \in A^{-1}\}$, where A^{-1} is the set of all invertible elements of A, then G has an interior point. For example the open neighborhood $V = \{f \in A : p_N(f) < 1/2\}$ of the origin is contained in G. Because if $f \in V$ then since the norm of a Banach function algebra is greater than the uniform norm we have $\|f\|_X \leq \|f|_{K_N}\|_N = p_N(f) < 1/2$ and so $(1 + f)(x) \neq 0$, for all $x \in X$ and since A is inverse closed $1 + f \in A$, that is $f \in G$.

Theorem 2.1. *Let X be a hemicompact space and $(A_n, \|.\|_n)$ and $(A, (p_n))$ be as above. Suppose that the algebra A separates the points of X and for each n, A_n is natural. If $(B, (q_n))$ is an Ff-algebra on X which contains A as a dense subalgebra and the identity map $I : (A, (p_n)) \longrightarrow (B, (q_n))$ is continuous then $M_A = M_B$ (as sets).*

Proof. For $n \in \mathbb{N}$ let $i(n)$ and p''_n and A_{K_n} be defined as in Theorem 1.3. We notice that here $i(n) = n$ and if $n \in \mathbb{N}$ and $f, g \in A$ and $f|_{K_n} = g|_{K_n}$ then $\|(f - g)|_{K_n}\|_n = p_n(f - g) = 0$. So $p_n(f) = p_n(g)$. This shows that $p''_n(f|_{K_n}) = p_n(f) = \|f|_{K_n}\|_n$, $f \in A$, and so A_{K_n} is indeed the closure of $A|_{K_n}$ in the Banach function algebra $(A_n, \|.\|_n)$. Therefore in this case each A_{K_n} is a Banach function algebra on K_n and $A = \varprojlim A_{K_n}$ by Theorem 1.3.

Now since I is a continuous monomorphism with a dense range $I^* : M_B \longrightarrow M_A$ defined by $I^*(\varphi) = \varphi|_A, \varphi \in M_B$, is an injective continuous map. Let $m \in \mathbb{N}$ and $f \in A$ then

$$\|\hat{f}\|_{M_{A_{K_m}}} = r_{A_{K_m}}(f|_{K_m}) = r_{A_m}(f|_{K_m}) = \|f\|_{K_m},$$

where $r_{A_m}(f|_{K_m})$ is the spectral radius of $f|_{K_m}$ in A_m and the last equality is a consequence of the naturality of A_m. On the other hand since $(B, (q_n))$ is an Ff-algebra on X for each $m \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ such that

$$\|f\|_{K_m} \leq \|\hat{f}\|_{M_{B_n}} \leq q_n(f) \quad (f \in B)$$

where B_n is the completion of $B/\ker q_n$ with respect to the norm $q'_n(f + \ker q_n) = q_n(f), f \in B$ (see inequality (1)). So by Theorem 1.4 I^* is also surjective (and proper). Therefore $M_A = M_B$ as sets.

Remarks. (a) In the above theorem if M_A is a k-space then since I^* is a proper map the restriction of ${I^*}^{-1}$ to each compact subset of M_A is continuous and so ${I^*}^{-1}$is continuous on M_A. Hence in this case M_A is homeomorhic to M_B.

(b) The naturality of A_n in the above theorem can not be omitted. For example let $X = [0,1]$ and $K_n = X$ and $A_n = A(\bar{D})|_{[-1,1]}$, $n \in \mathbb{N}$, where $\bar{D}$ is the closed unit disk in $\mathbb{C}$ and $A(\bar{D})$ is the uniform (Banach) algebra of continuous functions on $\bar{D}$ which are analytic on D. For each $f \in A_n$ there is a unique $g \in A(\bar{D})$ such that $g|_{[-1,1]} = f$. Define $\|f\|_n = \|g\|_{\bar{D}}$. Then clearly $A = \{f \in C(X) : f|_{K_n} \in A_n\} = A(\bar{D})|_{[-1,1]}$ and $M_A = \bar{D}$ and A is dense in $C([-1,1])$ but $M_{C([-1,1])} = [-1,1]$

Theorem 2.2 *Let X be a hemicompact non-compact space with (K_n) as an admissible exhaustion. Let $(A_n, \|.\|_n)$ and $(A, (p_n))$ be as in the beginning of this section and A separates the points of X. If B is a closed subalgebra of the Ff-algebra $(A, (p_n))$ then B can not be normable as a regular Banach algebra.*

Proof. Let $\|.\|$ be a norm on B such that $(B, \|.\|)$ is a regular Banach algebra on M_B. Since B is closed in A, $(B, (p_n))$ is a commutative semisimple Fréchet algebra. By the Carpenter's theorem (each commutative semisimple Fréchet algebra has a unique topology as a Fréchet algebra) [3] the identity map $I : (B, \|.\|) \longrightarrow (B, (p_n))$ is a homeomorphism. So there exists some $n_0 \in \mathbb{N}$ and a constant M such that

$$\|f\| \leq M.p_{n_0}(f) \quad (2)$$

holds for all $f \in B$.

Choose an $x \in X \backslash K_{n_0}$ (this is possible because X is non-compact). Then by the compactness of K_{n_0} in X and hence in M_B and the regularity of B on M_B there exists an $f \in B$ such that $\hat{f}(\varphi_x) = 1$ and $\hat{f}(\varphi_y) = 0$, for all $y \in K_{n_0}$, that is $f(x) = 1$ and $f|_{K_{n_0}} = 0$. Therefore $p_{n_0}(f) = 0$. Now inequality (2) implies that $\|f\| = 0$ and hence $f = 0$ as an element of B which is a contradiction.

Example 2.3. Let (X, d) be a metric space and let $0 < \alpha \leq 1$. The collection of all (complex) bounded Lipschitz functions of order α on X is denoted by $\mathrm{Lip}(X, \alpha)$. It is well-known that $\mathrm{Lip}(X, \alpha)$ with respect to pointwise multiplication is a Banach algebra under the norm $\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_X + p_\alpha(f) \qquad (f \in \mathrm{Lip}(X, \alpha)),$$

where $p_\alpha(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}$ and $\|f\|_X = \sup_{x \in X} |f(x)|$ [7].

Now let X be a hemicompact metric space and (K_n) an admissible exhaustion of X and let $0 < \alpha \leq 1$. Set $A_n = \mathrm{Lip}(K_n, \alpha)$ and

$$\|f\|_n = \|f\|_{K_n} + \sup_{\substack{x,y \in K_n \\ x \neq y}} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} \quad (f \in A_n).$$

Clearly $A_{n+1}|_{K_n} \subseteq A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$, $f \in A_{n+1}$. So by the above arguments $A = \{f \in C(X) : f|_{K_n} \in \mathrm{Lip}(K_n, \alpha), n \in \mathbb{N}\}$ is an Ff-algebra on X with respect to the topology defined by the sequence (p_n) of seminorms on it, where $p_n(f) = \|f|_{K_n}\|_n$, $n \in \mathbb{N}$ and $f \in A$. Notice that here A separates the points of X. Using Proposition 1.4 in [7] It is easy to see that A is dense in $C(X)$ in the compact-open topology. So by Theorem 2.1 $M_A = M_{C(X)} = X$.

Example 2.4. Let $0 < \alpha \leq 1$ and X be a perfect compact plane set which is a finite union of regular sets. By a *regular* set we mean a perfect compact plane set Y which is connected by rectifiable arcs and for each point $z_0 \in Y$ there exists a constant $c \geq 1$ such that $\delta(z, z_0) \leq c|z - z_0|$, for every $z \in Y$, in which δ is the geodesic metric on Y that is $\delta(z, z_0)$ is the infimum of the lengths of the rectifiable arcs joining z to z_0. Let $n \in \mathbb{N}$. The algebra of all functions f on X which are n-times differentiable and for each k, $0 \le k \le n$, $f^{(k)} \in C(X)$ ($f^{(k)} \in \mathrm{Lip}(X, \alpha)$) is denoted by $D^n(X)$ (resp. $\mathrm{Lip}^n(X, \alpha)$) and the algebra of all functions f with derivatives of all orders ($f^{(k)} \in \mathrm{Lip}(X, \alpha)$, for all k), is denoted by $D^\infty(X)$ (resp. $\mathrm{Lip}^\infty(X, \alpha)$).

It is well known that for each n, $D^n(X)$ and $\mathrm{Lip}^n(X, \alpha)$ are natural Banach function algebras on X under the norms defined by

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!}$$

and

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!}$$

respectively, where as before

$$p_\alpha(f^{(k)}) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{d^\alpha(x, y)},$$

(see [2] and [4]).

Now for each $n \in \mathbb{N}$ set $K_n = X$ and $A_n = D^n(X)$ ($A_n = \mathrm{Lip}^n(X, \alpha)$). Then in this case $A = \{f \in C(X) : f|_{K_n} \in A_n, n \in \mathbb{N}\} = D^\infty(X)$ ($A = \mathrm{Lip}^\infty(X, \alpha)$) $= \bigcap A_n$ and $(A, (\|.\|_n))$ is an Ff-algebra on X. Moreover we have the following inclusions:

$$R_0(X) \subseteq \mathrm{Lip}^\infty(X, \alpha) \subseteq \mathrm{Lip}^n(X, \alpha) \subseteq D^n(X) \subseteq D^1(X),$$

where $R_0(X)$ is the algebra of all rational functions with poles off X and $D^1(X) \subseteq R(X)$, the uniform closure of $R_0(X)$ [2]. Therefore A is dense in $R(X)$ and since each A_n is natural we have $M_A = M_{R(X)} = X$ by theorem 2.1. Indeed by the compactness of X, M_A is homeomorphic to X.

Remark. (a) It is interesting to notify that the algebra A defined in Example 2.3 is not in general a Banach algebra. Indeed it will be a Banach algebra iff X is compact.

(b) In Example 2.4 the algebras $\mathrm{Lip}^\infty(X, \alpha)$ and $D^\infty(X)$ are Q-algebras. Because in either case each A_n is inverse closed. Moreover there is no topology which makes these algebras to the Banach function algebras because $f \longmapsto f'$ defines a nontrivial derivation.

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A COMPARISON BETWEEN $C^*(N^+)$ AND $C_r^*(N^+)$

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ABSTRACT. By a well-known theorem in the theory of C^*-algebras (cf. [2, Theorem 7.7.7]) if G is an amenable group then $C^*(G) \cong C_r^*(G)$. Here we will prove that the above theorem is not valid for the semigroup N^+.

Introduction

In [4] we defined the C^*-algebra of $N^+, C^*(N^+)$; and that is the C^*-algebra generated by

$$V \oplus S \oplus S^* \oplus \bigoplus_{k=2}^{\infty} N_k.$$

In this paper we will determine the minimal ideals of $C^*(S \oplus S^*)$, and will prove that

$$C_r^*(N^+) \neq C^*(N^+).$$

Throughout this paper we assume that S is the unilateral shift operator and S^*, the adjoint of S is the backward shift operator.

We start with the following lemma.

1.1 Lemma. The C^*-algebra $C^*(S \oplus S^*)$ has a nontrivial closed two-sided ideal.

Proof. The following simple calculations show that

$$(K \oplus K) \cap C^*(S \oplus S^*)$$

is non empty.

$$(S \oplus S^*)^*(S \oplus S^*) - (S \oplus S^*)(S \oplus S^*)^* \in C^*(S \oplus S^*).$$

On the other hand by section 2.6 of [1] we have

$$(S \oplus S^*)^*(S \oplus S^*) - (S \oplus S^*)(S \oplus S^*)^* = (S^*S \oplus SS^*) - (SS^* \oplus S^*S)$$

$$= (S^*S - SS^*) \oplus (SS^* - S^*S),$$

since

$$(S^*S - SS^*) \oplus (SS^* - S^*S) = P_0 \oplus (-P_0) \in K \oplus K$$

where K is the ideal of all compact operators on $\ell^2(Z^+)$, we see that

$$P_0 \oplus (-P_0) \in (K \oplus K) \cap C^*(S \oplus S^*).$$

Therefore at least $(K \oplus K) \cap C^*(S \oplus S^*)$ is a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. $\square$

The following lemma is quite useful for our purpose.

1.2 Lemma. If J is a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$ then

$$P_0 \oplus 0 \in J \quad or \quad 0 \oplus P_0 \in J.$$

Proof. For $m \geq 0$ we have

$$(S \oplus S^*)^m(P_0 \oplus (-P_0))((S \oplus S^*)^*)^m = S^m P_0 (S^*)^m \oplus (S^*)^m(-P_0)S^m$$

$$= P_m \oplus 0 \in C^*(S \oplus S^*)$$

and

$$((S \oplus S^*)^*)^m(-P_0 \oplus P_0)(S \oplus S^*)^m = (S^*)^m(-P_0)S^m \oplus S^m P_0 (S^*)^m$$

$$= 0 \oplus P_m \in C^*(S \oplus S^*).$$

Now let J be a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. Hence there exists a non-zero element, say C, in J. Let $C = A \oplus B$. If $A \neq 0$, then for some $N \geq 0$ we have $A\delta_N \neq 0$. Since $A \oplus B \in J, P_N \oplus 0 \in C^*(S \oplus S^*)$ and J is an ideal we see that

$$(P_N \oplus 0)(A \oplus B)^*(A \oplus B)(P_N \oplus 0) = P_N A^* A P_N \oplus 0 \in J.$$

Now if $x \in \ell^2(Z^+)$, then

$$\begin{aligned} & (P_N A^* A P_N \oplus 0)(x \oplus 0) = P_N A^* A P_N x \\ = & \sum_{i=1}^{\infty} < P_N A^* Ax, \delta_i > \delta_i =< P_N A^* A\delta_N, \delta_N > \delta_N \\ = & < A\delta_N, A\delta_N > \delta_N = ||A\delta_N||^2 P_N x \\ = & ||A\delta_N||^2 (P_N \oplus 0)(x \oplus 0) \end{aligned}$$

i.e.,

$$P_N A^* A P_N \oplus 0 = P_N \oplus 0$$

Hence

$$P_N \oplus 0 \in J.$$

Thus

$$((S \oplus S^*)^*)^N (P_N \oplus 0)(S \oplus S^*)^N = (S^*)^N P_N S^N \oplus 0 = P_0 \oplus 0 \in J.$$

If $B \neq 0$ similar argument shows that

$$0 \oplus P_0 \in J.$$

This completes the proof. □

The following theorem determines the minimal ideals of $C^*(S \oplus S^*)$.

1.3 Theorem. $K \oplus 0$ and $0 \oplus K$ are two minimal closed two-sided ideals in $C^*(S \oplus S^*)$.

Proof. Let J be a nontrivial closed two-sided ideal in $C^*(S \oplus S^*)$. Since S and S^* have cyclic vectors by an argument similar to the proof of lemma 3 of [3] we see that J contains either all operators of the form $T_{y,z} \oplus 0$ or all operators of the form

$$0 \oplus T_{y,z}$$

where $y, z \in \ell^2(Z^+)$, and

$$T_{y,z}(x) =< x, y > z.$$

Since K is the norm closure of the ideal of all finite rank operators, we see that either $K \oplus 0 \subseteq J$ or $0 \oplus K \subseteq J$. Since J was arbitrary, the theorem is proved. □

Now it is time for making a comparison between the reduced C^*-algebra of N^+ and the full C^*-algebra of N^+ and get an important conclusion.

1.4 Corollary. $C_r^*(N^+) \neq C^*(N^+)$.

Proof. $C^*(N^+) = C^*(V \oplus S \oplus S^* \oplus \bigoplus_{k=2}^{\infty} N_k)$. Hence $C^*(N^+)$ is a C^*-subalgebra of $B(H)$ where $H = H_V \oplus H_S \oplus H_{S^*} \oplus \bigoplus_{k=2}^{\infty} H_{N_k}$. The inclusion mapping from $C^*(N^+)$ into $B(H)$ is a faithful representation of $C^*(N^+)$ on H. For each $A \in C^*(N^+)$, the mapping

$$A \to A|_{H_S \oplus H_{S^*}} : C^*(N^+) \to B(H_S \oplus H_{S^*})$$

is a representation of $C^*(N^+)$ on $H_S \oplus H_{S^*}$. By this representation the generator of $C^*(N^+)$ is mapped to $S \oplus S^*$. Hence

$$C^*(S \oplus S^*) \cong \text{a quotient of} C^*(N^+).(*)$$

By theorem 1.3 the left hand side of (*), and consequently $C^*(N^+)$ has at least two minimal ideals. Since by theorem 4 of [3], $C_r^*(N^+)$ has a unique minimal ideal, the proof is complete. □

We can summarize the result of this section in the following conclusion.

1.5 Conclusion. The well known theorem ([2, theorem 7.7.7]) which says that, if G is an amenable group, then

$$C_r^*(G) \cong C^*(G)$$

does not hold for the amenable semigroup N^+.

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AN APPLICATION OF THE ROSENTHAL-DOR THEOREM IN THE DIRICHLET SPACE

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ABSTRACT. Let G be a finitely connected domain and $D(G)$ be the Dirichlet space. If $\{w_n\}$ is a sequence in G such that $|w_n| \to \partial G$, then there exists subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that for a given sequence $\{a_k\}_k$ in ℓ^∞ we can find a function f in $D(G)$ satisfying $f(w_{n_k}) = a_k ||k_{w_{n_k}}||$.

INTRODUCTION

Let G be a domain in the complex plane. The Dirichlet space $D(G)$ is the Hilbert space of functions f analytic on G whose derivative f' is square integrable. Fixing a "base point" w in G, we define a norm for $D(G)$ by

$$||f||^2_{D(G)} = |f(w)|^2 + \int_G |f'|^2 dA.$$

The norms obtained by two different base points are equivalent. Let G be the open unit disk U and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in D(U)$, then

$$||f||^2_{D(U)} = |f(0)|^2 + \int_U |f'| dA = |a_0|^2 + \pi \sum_{n=1}^{\infty} n|a_n|^2.$$

So $D(U)$ is a subset of the Hardy space H^2 and the radial limit function of f is defined almost everywhere on ∂U.

It is well known that if λ is a point of G, then the linear functional of evaluation at $\lambda, e_\lambda : f \to f(\lambda)$ is norm continuous on $D(G)$. Hence to each point $\lambda \in G$ there corresponds a function k_λ in $D(G)$ such that $f(\lambda) =< f, k_\lambda >$ for all $f \in D(G)$. The function k_λ is called the reproducing kernel for the space $D(G)$ at the point λ. So convergence in $D(G)$ implies uniform convergence on compact subsets of G.

If φ is a function analytic on G with the property that $\varphi D(G) \subseteq D(G)$, then φ is called a multiplier for $D(G)$. The algebra of all multipliers for $D(G)$ is denoted by $M(D(G))$. Since $D(G)$ contains constants, $M(D(G)) \subset D(G)$. Every multiplier φ induce a linear transformation $M_\varphi : D(G) \longrightarrow D(G)$ defined by $M_\varphi f = \varphi f$ for all f in $D(G)$. By the closed graph theorem and the continuity of the linear functionals of evaluation at points in G, one can see that M_φ is a bounded linear operator and φ is a bounded function, with $||\varphi||_\infty \leq ||M_\varphi||$ where $||.||_\infty$ is the supremum norm on G. Thus $M(D(G)) \subseteq D(G) \cap H^\infty(G)$ where $H^\infty(G)$ denotes the algebra of all bounded analytic functions in G. Since the function z is in $M(D(U))$ thus every function analytic in $\bar{U}$ is a multiplier for $D(U)$. It is proved that if φ is a nonconstant function analytic on $\bar{U}$, then $||\varphi||_\infty < ||M_\varphi||$. So M_z can not be a contraction, since else it should be $||M_\varphi|| = ||\varphi||_\infty$ for all φ in $M(D(U))$.

Main Result

First we give the Rosenthal-Dor theorem that we need it for the proof of our main theorem.

Rosenthal-Dor Theorem: Suppose E is a Banach space and $\{e_n\}$ is a bounded sequence in E. Then there exists a subsequence $\{e_{n_k}\}$ such that either

i) the map $\{a_k\}_{k=1}^{\infty} \longrightarrow \sum\limits_{k=1}^{\infty} a_k e_{n_k}$ is an isomorphism of ℓ^1 into E,

or

ii) $\lim_k \varphi(e_{n_k})$ exists for every $\varphi \in E^*$.

Proof. See [3] and [4].

Theorem. Suppose $\{w_n\}$ is a sequence in U such that $|w_n| \longrightarrow 1$. Then for some subsequence $\{w_{n_k}\}$ we have $\ell^\infty = \{\{\frac{f(w_{n_k})}{||k_{w_{n_k}}||}\}_k : f \in D(U)\}$.

Proof. Put $e_n = \frac{k_{w_n}}{||k_{w_n}||}$ for all $n \in \mathbf{N}$. Then $\{e_n\}_n$ is a bounded sequence in $D(U)$. Let $\{e_{n_k}\}$ promised by the Rosenthal-Dor theorem, and suppose that case (i) of the theorem holds. Let T denote the isomorphism from ℓ^1 into $D(G)$ given by case (i) of the Rosenthal-Dor theorem. Because T is one to one and has closed range, the dual T^* maps $D(U)^*$ onto ℓ^∞. By the Riesze Representation theorem $D(U)^* = D(U)$. Indeed $D(U)^* = \{L_f : f \in D(U)\}$ where $L_f : D(U) \longrightarrow \mathbf{C}$ is defined by $L_f(g) =< g, f >$ for all g in $D(U)$. Now let $a = \{a_n\} \in \ell^\infty$. So $\bar{a} = \{\bar{a}_n\}_n \in \ell^\infty$. Since T^* is onto, there exists $L_f \in D(U)^*$ such that $T^* L_f = \bar{a}$. Recall that $T^* L_f = L_f \circ T$. So $L_f \circ T = \bar{a}$. Apply both sides of the equation $L_f \circ T = \bar{a}$ to the vector in ℓ^1 that is 0 except for a 1 in the kth coordinate, getting $L_f e_{n_k} = \bar{a}_k$ for every k. Thus

$$\begin{aligned} \bar{a}_k &= L_f e_{n_k} =< e_{n_k}, f > \\ &= < \frac{k_{w_{n_k}}}{||k_{w_{n_k}}||}, f >= \frac{\overline{f(w_{n_k})}}{||k_{wn_k}||} \end{aligned}$$

for all k and so $\frac{f(w_{n_k})}{||k_{w_{n_k}}||} = a_k$ for all k. Therefore

$$\ell^\infty \subseteq \{\{\frac{f(w_{n_k})}{||k_{w_{n_k}}||}\}_k : f \in D(U)\},$$

if we can prove that case (ii) of the Rosenthal-Dor theorem never holds. We will find h in $D(U)$ such that $\lim_k L_h(e_{n_k})$ does not exist: By proposition 4 of [1], there exists $\varphi \in M(D(U))$ such that $\varphi(w_{n_k}) = (-1)^k$ and so $\lim_k \varphi(w_{n_k})$ does not exist. For suitable choices of $\theta_k, e^{-i\theta_k w_{n_k}}$ is a positive real number for all k. Now consider the sequence $\{a_n\}$ of positive real numbers such that $\sum_n na_n^2$ is finite. Put $\psi(z) = \sum_{k=0}^{\infty} a_k e^{-i\theta_k z^k}$. Then $\psi \in D(U)$ and $\psi(w_{n_k})$ is a positive real number. Define $h = \varphi\psi$. Since $\varphi \in M(D(U)), h = \varphi\psi \in D(U)$ and we have:

$$\begin{aligned} L_n(e_{n_k}) &= \frac{h(w_{n_k})}{||k_{w_{n_k}}||} = \frac{\varphi(w_{n_k})\psi(w_{n_k})}{||k_{w_{n_k}}||} \\ &= (-1)^k \frac{\psi(w_{n_k})}{||k_{w_{n_k}}||} \ . \end{aligned}$$

But

$$0 \leq \frac{\psi(w_{n_k})}{||k_{w_{n_k}}||} = \frac{< \psi, k_{w_{n_k}} >}{||k_{w_{n_k}}||} \leq ||\psi|| \ \ (\forall k) \ \ .$$

So $\lim_k L_n(e_{n_k})$ does not exist. This completes the proof.

Consider the circular domain $G = U \backslash K_1 \cup \cdots \cup K_N$ where $K_1 = \bar{D}_i = \{z : |z - z_i| \leq r_i\}$ $(i = 1, \ldots, N)$ are disjoint closed subdisks of the open unit disk U. Put $G_i = \mathbf{C} \cup \{\infty\} \backslash K_i$ $(i = 1, \ldots, N)$. Then it is proved that:

$$D(G) = D(U) + D_0(G_1) + \cdots + D_0(G_N)$$

where $D_0(G_i) = D(G_i) \cap H_0(G_i)$ and $H_0(G_i)$ denotes the space of all functions in $H(G_i)$ that vanishes at ∞ ([2]).

The above theorem can be extended for the case of circular domain instead of the unit disk:

Theorem. Let $\{w_n\}$ be a sequence in the circular domain G such that $|w_n| \to r_i$ for some $0 \leq i \leq N$ $(G_0 = U)$. Then there exists a subsequence $\{w_{n_k}\}$ such that

$$\ell^\infty = \{\{\frac{f(w_{n_k})}{||k_{w_{n_k}}||}\}_k : f \in D(G)\}.$$

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CYCLICITY ON A SPECIAL BANACH SPACE

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Abstract. Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers and $1 \leq p < \infty$. We consider the space $\ell^p(\beta)$ of all power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p < \infty$. We give a necessary and sufficient condition for a polynomial to be cyclic in $\ell^p(\beta)$.

Introduction

First in the following, we generalize the definition coming in [1].

Let $\{\beta(n)\}$ be a sequence of nonzero complex numbers with $\beta(0) = 1$ and $1 \leq p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$||f||^p = ||f||_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p |\beta(n)|^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $\ell^p(\beta)$ denote the space of such formal power series. These are Banach spaces with the norm $||.||_\beta$.

Let $\hat{f}_k(n) = \delta_{nk}$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $||f_k|| = |\beta(k)|$. Now consider M_z, the operator of multiplication by z on $\ell^p(\beta)$:

$$(M_z f)(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^{n+1}.$$

In otherwords

$$(M_z\hat{f})(n) = \begin{cases} \hat{f}(n-1) & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case

$$||M_z^m|| = \sup_k |\frac{\beta(n+k)}{\beta(k)}|, n = 0, 1, 2, \cdots .$$

Consider the multiplication of formal power series, $fg = h$ given by

$$\left(\sum_{n=0}^{\infty} \hat{f}(n)z^n\right) . \left(\sum_{n=0}^{\infty} \hat{g}(n)z^n\right) = \sum_{n=0}^{\infty} \hat{h}(n)z^n$$

where

$$\hat{h}(n) = \sum_{k=0}^{n} \hat{f}(k)\hat{g}(n-k), n = 0, 1, 2, \cdots$$

If $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\sup_n \sum_{i=1}^{n} |\frac{\beta(n)}{\beta(i)\beta(n-i)}|^q < \infty$$

then clearly by the Holder's inequality one can see that $\ell^p(\beta)$ is a Banach algebra.

If $f \in \ell^p(\beta)$ and $P(z)$ is a polynomial, then to the vector $P(M_z)f$ there corresponds the formal power series $P(z)f(z)$.

Let X be a Banach space. We denote by $B(X)$, the set of bounded operators on the Banach space X. Let $A \in B(X)$ and $x \in X$. We say that x is a cyclic vector of A if

$$X = span\{A^n x : n = 0, 1, 2, \cdots\}.$$

Here $span\{.\}$ is the closed linear span of the set $\{.\}$. We investigate the cyclicity of a polynomial in $\ell^p(\beta)$.

Main Result

Remember that if λ is a complex number, then e_λ denotes the functional of "evaluation at λ", defined on polynomials by

$$e_\lambda(P) = P(\lambda),$$

and also λ is said to be a bounded point evaluation on $\ell^p(\beta)$ if the functional e_λ extends to a bounded linear functional on $\ell^p(\beta)$. In this case we use $f(\lambda)$ to denote $e_\lambda(f)$ for f in $\ell^p(\beta)$, see [1] for the case $p = 2$. Since the polynomials are dense in $\ell^p(\beta)$, this is equivalent to the existence of a constant $c > 0$ such that $|e_\lambda(P)| \leq c||P||_p$ for all polynomials P.

For $1 < p < \infty, \ell^p(\beta) \cong L^p(\mu)$ where μ is the σ-finite measure defined on the positive integers by $\mu(K) = \sum\limits_{n\in k}(\beta(n))^p, K \subseteq N \cap \{0\}$. So $\ell^p(\beta)$ is a reflexive Banach space.

Theorem 1. The zeros of a polynomial P are not bounded point evaluations on $\ell^p(\beta)$ if and only if P is cyclic.

Proof. Assume that $P(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$ and $\lambda_i's$ are not bounded point evaluations on $\ell^p(\beta)$. Let $1 \leq i \leq m$ and $L \in (\ell^p(\beta))^*$ be such that

$$L(z^n(z-\lambda_i)) = 0, \quad n = 0, 1, 2, \cdots$$

Since $(\ell^p(\beta))^* = \ell^q(\beta^{p/q})$ (see[2]), there exists unique element $g \in \ell^q(\beta^{p/q})$ such that for all $f \in \ell^p(\beta)$ we have

$$L\left(\sum_n \hat{f}(n)z^n\right) = \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)\overline{(\beta(n))}^{p/q}.$$

Now we have

$$\begin{aligned} 0 = L(z^{n-1}(z-\lambda_i)) &= L(z^n - \lambda_i z^{n-1}) \\ &= \overline{\hat{g}(n)}\beta(n)\overline{(\beta(n))}^{p/q} \\ &- \lambda_i\overline{\hat{g}(n-1)}\beta(n-1)\overline{(\beta(n-1))}^{p/q} \end{aligned}$$

for $n = 1, 2, \cdots$ thus

$$\begin{aligned} \overline{\hat{g}(n)} &= \lambda_i \frac{\beta(n-1)\overline{\beta(n-1))}^{p/q}}{\beta(n)\overline{\beta(n))}^{p/q}}.\overline{\hat{g}(n-1)} \\ &= \lambda_i^2 \frac{\beta(n-2)\overline{(\beta(n-2))}^{p/q}}{\beta(n)\overline{(\beta(n))}^{p/q}}.\overline{\hat{g}(n-2)} \\ &= \frac{\lambda_i^n}{\beta(n)\overline{(\beta(n))}^{p/q}}.\overline{\hat{g}(0)}. \end{aligned}$$

Therefore

$$\overline{\hat{g}(n)(\beta(n))}^{p/q} = \frac{\lambda_i^n}{\beta(n)}.\overline{\hat{g}(0)}$$

for $n = 1, 2, \cdots$. By part (i) of [2], $\left\{\frac{\lambda_i^n}{\beta(n)}\right\}_n \notin \ell^q$. But since $g \in \ell^q(\beta^{p/q})$, we have

$$\{\hat{g}(n)(\beta(n))^{p/q}\}_n \in \ell^q.$$

This implies that it should be

$$\hat{g}(n) = 0, \quad n = 0, 1, 2, \cdots$$

So $L = 0$. Now by the Hahn Banach Theorem $z - \lambda_i$ is cyclic for $i = 1, 2, \cdots, m$ and therefore $P(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$ is cyclic in $\ell^p(\beta)$.

Conversely assume that P is cyclic, and choose polynomials $\{P_n\}$ such that $P_n f \to 1$ in $\ell^p(\beta)$. Now if $f(\lambda) = 0$ and λ is a bounded point evaluation on $\ell^p(\beta)$, then $e_\lambda(P_n f) \to 1$, that is a contradiction. This completes the proof.

Let $r_0 = \underline{\lim}(\beta(n))^{1/n} \neq 0$ and $\Omega = \{z : |z| < r_0\}$. Put

$$\ell_a^p(\beta) = \{f|_\Omega : f \in \ell^p(\beta)\}.$$

Clearly the functions of $\ell_a^p(\beta)$ are analytic. We denote by $\ell_a^\infty(\beta)$ the set of all formal power series $\varphi(z) = \sum_n \hat{\varphi}(n)z^n$ such that $\varphi\ell_a^p(\beta) \subseteq \ell_a^p(\beta)$.

Theorem 2. Suppose φ is in $\ell_a^\infty(\beta)$. If $\varphi(\Omega)$ intersects the unit circle, then $\{(M_\varphi^*)^n\}_n$ converges pointwise to zero on a dense subset of $\ell_a^p(\beta)$ and also there is a dense subset X of $\ell_a^p(\beta)$ and a map $S : X \to X$ such that $M_\varphi^* S$ is identity of X and $\{S^n\}_{n=0}^\infty$ tends pointwise to zero on X.

Proof. Suppose $\varphi(\Omega)$ intersects the unit circle. Since φ is nonconstant, by the open mapping theorem $\varphi(\Omega)$ is open. So the open sets

$$V = \{z \in \Omega : |\varphi(z)| < 1\} \ \textit{and} \ W = \{z \in \Omega : |\varphi(z)| > 1\}$$

are both nonempty. Now clearly the linear subspaces

$$H_V = V\{e_\lambda : \lambda \in V\}, \qquad H_W = \{e_\lambda : \lambda \in W\}$$

are dense in $\ell^q(\beta^{p/q})$. Since $(M_\varphi^*)^n = (M_\varphi^n)^*$ is the adjoint of multiplication by φ^n,

$$(M_\varphi^*)^n e_\lambda = (\varphi(\lambda))^n \, e_\lambda, \qquad n = 0, 1, 2, \cdots$$

If $z \in V$, so $|\varphi(z)| < 1$ and so $|\varphi(\lambda)|^n.||e_\lambda|| \to 0$ as $n \to \infty$. Thus $||(M_\varphi^*)^n e_\lambda|| \to 0$ as $n \to \infty$ and so the sequence of operators $\{(M_\varphi^*)^n\}$ converges pointwise to zero on the dense subset H_V spanned by the kernels $\{e_\lambda : \lambda \in V\}$.

Now we want to find a right inverse of M_φ. First consider the special case where the collection of reproducing kernels $\{e_\lambda : \lambda \in W\}$ is linearly independent. In this case we can define a linear map

$S : H_W \to H_W$ by extending the definition $Se_\lambda = (\varphi(\lambda))^{-1}e_\lambda \quad (\lambda \in W)$ linearly to H_w. Since $|\varphi(z)| > 1$ for each $z \in W$, these is no possibility of dividing by zero, and moreover,

$$S^n e_\lambda = (\varphi(\lambda))^{-1}e_\lambda \to 0 \quad in \quad \ell^q(\beta^{p/q}) \quad as \quad n \to \infty.$$

By definition $M_\varphi^* S=$ identity on the dense subset H_W of $\ell^q(\beta^{p/q})$.

In case the reproducing kernels are not linearly independent, a little more care is required. We can find an infinite subset $Z = \{w_n\}$ of W for which the corresponding set of kernel functions is linearly independent, and spans a subspace H_Z which is dense in $\ell^q(\beta^{p/q})$. The operator S can now be defined exactly as in the last paragraph, with H_Z in place of H_W. This completes the proof.

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SPACES OF COMPACT OPERATORS WITHOUT THE FIXED POINT PROPERTY

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ABSTRACT. Let X and Y are two infinite dimensional Banach spaces. If either X^* or Y contains an asymptotically isometric copy of c_0, then $K(X,Y)$, the space of all compact operators fails the fixed point property.

Dowling, Lennard and Turett [3] introduced the concept of Banach spaces containing an asymptotically isometric copy of c_0 . This notion was motivated by a James's distortion theorem [5].
This concept is a tool in identifying Banach spaces failing fixed point property. In fact they have shown that If X contains an asymptotically isometric copy of c_0, then X fails the Fixed Point Property(FPP). A Banach space X has the Fixed Point Property, if given any nonempty closed bounded and convex subset C of X, every nonexpansive mapping $T : C \to C$ has a fixed point. T is nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|, \ \forall x, y \in C.$$

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Definition. [3]. *Banach space X contains an asymptotically isometric copy of c_0 if there is a null sequence (ε_n) in $(0,1)$ and a sequence(x_n) in X so that for all finite sequence of scalars $(a_n)_{n=1}^{n=N}$,*

$$\max_{1\le n\le N}(1-\varepsilon_n)|a_n| \le \|\sum_{n=1}^{n=N} a_n x_n\| \le \max_{1\le n\le N}|a_n|.$$

Dowling, Lennard and Turett in [4] obtained an analogous result of E. Saab and P. Saab [7] in injective tensor products for the containment of an asymptotically isometric copy of c_0. In the following result we give a simple improvement of the theorem 8 of [4] for $K_{w^*}(X^*, Y)$, the space of all compact operators from X^* into Y which are $w^* - w$ continuous provide with usual operator norm. It is well known that the injective tensor product of X and Y, $X\widehat{\otimes}_\epsilon Y$ is a subspace of $K_{w^*}(X^*, Y)$. Furthermore the spaces of all compact operators $K(X,Y)$ is isometrically isomorphic to $K_{w^*}(X^{**}, Y)$ [6].

Theorem. *Let X and Y be two infinite dimensional Banach spaces. If X contains an asymptotically isometric copy of c_0, then $X\widehat{\otimes}_\epsilon Y$ contains an asymptotically isometric copy of c_0 which is complemented in $K_{w^*}(X^*, Y)$.*

Proof. Without loss of generality, there are a null sequence (ε_n) in $(0, 1/2)$ and a sequence (x_n) in X such that for all finite sequence of scalars $(a_n)_{n=1}^{n=N}$,

$$\max_{1\le n\le N}(1-\varepsilon_n)|a_n| \le \sum_{n=1}^{n=N} a_n x_n\| \le \max_{1\le n\le N}|a_n|$$

for all natural number N. Let (x_n^*) be a sequence of Hahn Banach extensions of the coordinate functionals associated with the sequence (x_n). Thus $\|x_n^*\| = 1$ and $x_m^*(x_n) = 0$ if $m \neq n$. By Josefson-Nissenzweig Theorem [2], there exists a normalized w^*-null sequence (y_n^*) in Y^*. Choose $y_n \in Y$ such that $\|y_n\| = 1$ and $y_n^*(y_n) \geq (1 - \varepsilon_n)$ for every n. We claim that the sequence $(x_n \otimes y_n)$ is an asymptotically isometric c_0-sequence in $X \widehat{\otimes}_\varepsilon Y$. First

$$\|\sum_{n=1}^{n=N} a_n x_n \otimes y_n\| = \sup_{y^* \in B_Y^*} \|\sum_{n=1}^{n=N} a_n y^*(y_n) x_n\|$$

$$\leq \sup_{y^* \in B_Y^*} \max_{1 \leq n \leq N} |a_n| \, |y^*(y_n)| = \max_{1 \leq n \leq N} |a_n|.$$

On the other hand, for $1 \leq m \leq N$

$$\|\sum_{n=1}^{n=N} a_n(x_n \otimes y_n)\| \geq |(x_m^* \otimes y_m^*)(\sum_{n=1}^{n=N} a_n(x_n \otimes y_n))| \geq (1 - \varepsilon_m)|a_m|.$$

Hence $(x_n \otimes y_n)$ is an asymptotically isometric c_0-sequence in $X \widehat{\otimes}_\varepsilon Y \subset K_{w^*}(X^*, Y)$. As (y_n^*) is a *weak** null sequence in Y^*, therefore the sequence $(x_n^* \otimes y_n^*)$ is *weak** null in $K_{w^*}(X^*, Y)$. Moreover $(x_n^* \otimes y_n^*)(x_n \otimes y_n) \geq 1/2$. Now an appeal to the Theorem 1.1.2 of [1] completes the proof.

As an application of this Theorem we obtain the following result for FPP of the space of compact operators.

Corollary. *Let X and Y be two infinite dimensional Banach spaces. If either X^* or Y contains an asymptotically isometric copy of c_0, then $K(X, Y)$ fails FPP.*

Proof. As $K(X, Y)$ is isometrically isomorphic to $K_{w^*}(X^{**}, Y)$, by Theorem , $K(X, Y)$ has an asymptotically isometric complemented copy of

c_0 in the first case. For the second case, we observe that the canonical embedding of $Y\widehat{\otimes}_\varepsilon X^* = X^*\widehat{\otimes}_\varepsilon Y$ into $K(X,Y)$, followed by the mapping $u \to u^t$, yields the canonical embedding of $X^*\widehat{\otimes}_\varepsilon Y$ into $K(Y^*, X^*)$. Hence we can apply the first case to find an asymptotically isometric copy of c_0 in $X^*\widehat{\otimes}_\varepsilon Y$ that is complemented in $K(Y^*, X^*)$, and therefore is complemented in the intermediate space $K(X,Y)$. Now an appeal to the Proposition 7 of [3] completes the proof.

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