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Delaunay Triangulations-Part II

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December 2014
Purpose: Approximating a terrain by constructing a polyhedral terrain from a set $P$ of sample points.
Theorem 9.1:
Let \( P = \{ p_1, p_2, \ldots, p_n \} \) be a point set. A triangulation of \( P \) is a maximal planar subdivision with vertex set \( P \).

\[ \text{triangles} = 2n - 2 - k \]
\[ \text{edges} = 3n - 3 - k \]

where \( k \) is the number of points in \( P \) on the convex hull of \( P \).

Theorem 9.2: (Thales Theorem)
Let \( C \) be a circle, \( L \) a line intersecting \( C \) in points \( a \) and \( b \), and \( p, q, r \) and \( s \) points lying on the same side of \( L \). Suppose that \( p \) and \( q \) lie on \( C \), that \( r \) lies inside \( C \), and that \( s \) lies outside \( C \). Then

\[ \angle arb > \angle abq = \angle aqb > \angle asb \]
Observation 9.3:

Let $T$ be a triangulation with an illegal edge $e$. Let $T'$ be the triangulation obtained from $T$ by flipping $e$. Then

$$A(T') > A(T)$$

Definition:

A legal triangulation is a triangulation that does not contain any illegal edge.

Conclusion:

Any angle-optimal triangulation is legal.
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Lemma 9.4:
Let edge $p_i p_j$ be incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$ and let $C$ be the circle through $p_i, p_j$ and $p_k$. The edge $p_i p_j$ is illegal if and only if the point $p_l$ lies in the interior of $C$.

*⇒* if the points $p_i, p_j, p_k, p_l$ form a convex quadrilateral and do not lie on a common circle ⇒ exactly one of $p_i p_j$ and $p_k p_l$ is an illegal edge.
Lemma 9.4:

Let edge $p_ip_j$ be incident to triangles $p_ip_jp_k$ and $p_ip_jp_l$ and let $C$ be the circle through $p_i, p_j$ and $p_k$. The edge $p_ip_j$ is illegal if and only if the point $p_l$ lies in the interior of $C$.

If the points $p_i, p_j, p_k, p_l$ form a convex quadrilateral and do not lie on a common circle $\Rightarrow$ exactly one of $p_ip_j$ and $p_kp_l$ is an illegal edge.
Lemma 9.4:

Let edge $p_i p_j$ be incident to triangles $p_i p_j p_k$ and $p_i p_j p_l$ and let $C$ be the circle through $p_i, p_j$ and $p_k$. The edge $p_i p_j$ is illegal if and only if the point $p_l$ lies in the interior of $C$.

If the points $p_i, p_j, p_k, p_l$ from a convex quadrilateral and do not lie on a common circle $⇒$ exactly one of $p_i p_j$ and $p_k p_l$ is an illegal edge.
**Algorithm** `LEGAL_TRIANGULATION(\mathcal{T})`

*Input.* A triangulation \(\mathcal{T}\) of a point set \(P\).

*Output.* A legal triangulation of \(P\).

1. while \(\mathcal{T}\) contains an illegal edge \(\overline{p_ip_j}\)
2. do (* Flip \(\overline{p_ip_j}\) *)
3. Let \(\overline{p_ip_jp_k}\) and \(\overline{p_ip_jp_l}\) be the two triangles adjacent to \(\overline{p_ip_j}\).
4. Remove \(\overline{p_ip_j}\) from \(\mathcal{T}\), and add \(\overline{p_kp_l}\) instead.
5. return \(\mathcal{T}\)
A set $P$ of $n$ points in the plane

2. The Voronoi diagram $V_{or}(P)$ is the subdivision of the plane into Voronoi cells $V(p)$ for all $p \in P$

3. Let $G$ be the dual graph of $V_{or}(P)$

4. The Delaunay graph $DG(P)$ is the straight line embedding of $G$. 
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Back
A set $P$ of $n$ points in the plane

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3. Let $G$ be the dual graph of $Vor(P)$

4. The Delaunay graph $DG(P)$ is the straight line embedding of $G$. 
Lemma 9.5:
The Delaunay graph of a planar point set is a plane graph.

The edge $p_ip_j$ is in the Delaunay graph $Dg(P) \iff$ there is a $C_{ij}$ with $p_i$ and $p_j$ on its boundary and no other site of $P$ contained in it.
The center of such a disc lies on the common edge of $V(p_i)$ and $V(p_j)$.

If the point set $P$ is in general position then the Delaunay graph is a triangulation.
Theorem 9.6:
Let $P$ be a set of points in the plane,
- Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the Delaunay graph of $P$ $\iff$ the circle through $p_i, p_j, p_k$ contains no point of $P$ in its interior.
- Two points $p_i, p_j \in P$ form an edge of the Delaunay graph of $P$ $\iff$ there is a closed disc $C$ that contains $p_i$ and $p_j$ on its boundary and does not contain any other point of $P$.

Theorem 9.7:
$T$ is a Delaunay triangulation of $P$ $\iff$ the circumcircle of any triangle of $T$ does not contain a point of $P$ in its interior.
Review

Computing the Delaunay Triangulation

Analysis

Conclusion

Triangulations of Planar Point Sets

The Delaunay Triangulation

Voronoi

Delaunay

Empty sphere
Theorem 9.8:
Let $P$ be a set of points in the plane. A triangulation $T$ of $P$ is legal $\iff T$ is a Delaunay triangulation $P$.

Theorem 9.9:
Let $P$ be a set of points in the plane. Any angle-optimal triangulation of $P$ is a Delaunay triangulation $P$. Furthermore, any Delaunay triangulation of $P$ maximizes the minimum angle over all triangulations of $P$. 
A Delaunay triangulation for a set $P$ of points in a plane is a triangulation $DT(P)$ such that no point in $P$ is inside the circumcircle of any triangle in $DT(P)$. 

Diagram: A Delaunay triangulation of a set of points.
Delaunay triangulations maximize the minimum angle of all the angles of the triangles in the triangulation; they tend to avoid skinny triangles.

For a set of points on the same line there is no Delaunay triangulation (the notion of triangulation is degenerate for this case).

For four or more points on the same circle (e.g., the vertices of a rectangle) the Delaunay triangulation is not unique.

By considering circumscribed spheres, the notion of Delaunay triangulation extends to three and higher dimensions.
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By considering circumscribed spheres, the notion of Delaunay triangulation extends to three and higher dimensions.
The triangulation is named after Boris Delaunay for his work on this topic from 1934.
The voronoi diagram is named after Georgy F. Voronoi for his work on this topic.
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Review
Usage

Delaunay triangulations help in constructing various things:

- Euclidean Minimum Spanning Trees
- Approximations to the Euclidean Traveling Salesperson Problem
- ...
Reconstruction
Usage

Meshing
Usage

Meshing / Remeshing
Methods

There are several ways to compute the Delaunay triangulation:

- By plane sweep
- By iterative flipping from any triangulation
- By conversion from the Voronoi diagram
- By randomized incremental approach
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![Diagram of triangles and points](image-url)
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P_0
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- P -1

![Diagram of Delaunay Triangulation](image)

Points labeled P0, P-1, and P-2 connected by lines forming a triangulation.
Randomized incremental approach
Randomized incremental approach
Randomized incremental approach
Randomized incremental approach
Algorithm `DELAUNAY_TRIANGULATION(P)`

Input. A set $P$ of $n+1$ points in the plane.

Output. A Delaunay triangulation of $P$.

1. Let $p_0$ be the lexicographically highest point of $P$, that is, the rightmost among the points with largest $y$-coordinate.
2. Let $p_{-1}$ and $p_{-2}$ be two points in $\mathbb{R}^2$ sufficiently far away and such that $P$ is contained in the triangle $p_0p_{-1}p_{-2}$.
3. Initialize $\mathcal{T}$ as the triangulation consisting of the single triangle $p_0p_{-1}p_{-2}$.
4. Compute a random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$.
5. for $r \leftarrow 1$ to $n$
   6. do (* Insert $p_r$ into $\mathcal{T}$: *)
   7. Find a triangle $p_ip_jp_k \in \mathcal{T}$ containing $p_r$.
   8. if $p_r$ lies in the interior of the triangle $p_ip_jp_k$
   9. then Add edges from $p_r$ to the three vertices of $p_ip_jp_k$, thereby splitting $p_ip_jp_k$ into three triangles.
10. \textsc{LEGALIZEEDGE}(p_r, p_ip_j, \mathcal{T})
11. \textsc{LEGALIZEEDGE}(p_r, p_jp_k, \mathcal{T})
12. \textsc{LEGALIZEEDGE}(p_r, p_kp_i, \mathcal{T})
Algorithm \textsc{DelaunayTriangulation}(P)

\textit{Input.} A set \( P \) of \( n + 1 \) points in the plane.

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1. Let \( p_0 \) be the lexicographically highest point of \( P \), that is, the rightmost among the points with largest \( y \)-coordinate.
2. Let \( p_{-1} \) and \( p_{-2} \) be two points in \( \mathbb{R}^2 \) sufficiently far away and such that \( P \) is contained in the triangle \( p_0 p_{-1} p_{-2} \).
3. Initialize \( \mathcal{T} \) as the triangulation consisting of the single triangle \( p_0 p_{-1} p_{-2} \).
4. Compute a random permutation \( p_1, p_2, \ldots, p_n \) of \( P \setminus \{ p_0 \} \).
5. \textbf{for} \( r \leftarrow 1 \) \textbf{to} \( n \)
6. \hspace{1em} \textbf{do} (* Insert \( p_r \) into \( \mathcal{T} \): *)
7. \hspace{2em} Find a triangle \( p_i p_j p_k \in \mathcal{T} \) containing \( p_r \).
8. \hspace{2em} \textbf{if} \( p_r \) lies in the interior of the triangle \( p_i p_j p_k \)
9. \hspace{2em} \hspace{1em} \textbf{then} Add edges from \( p_r \) to the three vertices of \( p_i p_j p_k \), thereby splitting \( p_i p_j p_k \) into three triangles.
10. \hspace{2em} \textsc{LegalizeEdge}(p_r, p_i p_j, \mathcal{T})
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12. \hspace{2em} \textsc{LegalizeEdge}(p_r, p_k p_i, \mathcal{T})
Randomized incremental approach

\( p_r \) lies in the interior of a triangle

\( p_r \) falls on an edge
13. else (**p_r** lies on an edge of **p_ip_jp_k**, say the edge **p_ip_j**)  
14. Add edges from **p_r** to **p_k** and to the third vertex **p_l** of the other triangle that is incident to **p_ip_j**, thereby splitting the two triangles incident to **p_ip_j** into four triangles.  
15. `LEGALIZEEDGE(p_r, p_ip_l, T)`  
16. `LEGALIZEEDGE(p_r, p_ip_j, T)`  
17. `LEGALIZEEDGE(p_r, p_jp_k, T)`  
18. `LEGALIZEEDGE(p_r, p_kp_i, T)`  
19. Discard **p_1** and **p_2** with all their incident edges from **T**.  
20. return **T**
Randomized incremental approach

Lemma 9.4

\[
\text{LEGALIZEEDGE}(p_r, \overline{p_ip_j}, \mathcal{J})
\]
1. (* The point being inserted is \( p_r \), and \( \overline{p_ip_j} \) is the edge of \( \mathcal{J} \) that may need to be flipped. *)
2. \( \text{if} \ \overline{p_ip_j} \ \text{is illegal} \)
3. \( \text{then} \ \text{Let} \ \overline{p_ip_j}p_k \ \text{be the triangle adjacent to} \ \overline{p_ip_j}p \ \text{along} \ \overline{p_ip_j}. \)
4. (* Flip \( \overline{p_ip_j} \): *) Replace \( \overline{p_ip_j} \) with \( \overline{p_ip_k} \).
5. \( \text{LEGALIZEEDGE}(p_r, \overline{p_ip_k}, \mathcal{J}) \)
6. \( \text{LEGALIZEEDGE}(p_r, \overline{p_kp_j}, \mathcal{J}) \)
Randomized incremental approach

But

what about the correctness of algorithm?
Randomized incremental approach

Must show no illegal edge left behind!

- We see that every new edge added is incident to $P_r$.
- We will see that every new edge added is in fact legal.
- Together with the fact that an edge can only become illegal if one of its incident triangles changes, then our algorithm tests any edge that may become illegal.
Randomized incremental approach

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The algorithm is correct.
Randomized incremental approach

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The algorithm is correct
Randomized incremental approach

Lemma 9.10:
Every new edge created in ’DELAUNAYTRIANGULATION’ or in ’LEGALIZEEDGE’ during the insertion of $P_r$ is an edge of the Delaunay graph of $\{p_{-1}, p_{-2}, p_0, \ldots, p_r\}$
Randomized incremental approach

Lemma 9.10:

Every new edge created in ‘DELAUNAYTRIANGULATION’ or in ‘LEGALIZEEDGE’ during the insertion of $P_r$ is an edge of the Delaunay graph of $\{p_{r-1}, p_{r-2}, p_0, \ldots, p_r\}$
Randomized incremental approach

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How to find the triangle containing the point $p_r$?
A point location structure $D$ is a directed acyclic graph.

- The leaves of $D$ correspond to the triangles of the current triangulation $T$
- exist cross-pointers between those leaves and the triangulation.
- The internal nodes of $D$ correspond to triangles that have already been destroyed
- Any internal node gets at most three outgoing pointers.
A point location structure D is a directed acyclic graph.

- The leaves of D correspond to the triangles of the current triangulation T
  - exist cross-pointers between those leaves and the triangulation.
- The internal nodes of D correspond to triangles that have already been destroyed
  - Any internal node gets at most three outgoing pointers.
Constructing DAG

\[ P_{-1} \rightarrow P_0 \rightarrow \Delta_2 \rightarrow \Delta_3 \rightarrow \Delta_4 \rightarrow \Delta_1 \]

\[ P_{-2} \]

\[ \Delta_1 \rightarrow \Delta_2 \rightarrow \Delta_3 \rightarrow \Delta_4 \]

\[ P_0 \]

\[ \Delta_2, \Delta_3, \Delta_4 \]
Constructing DAG
Constructing DAG

Diagram showing the Delaunay triangulation with point locations and edges connecting the triangles in a directed acyclic graph (DAG).
Constructing DAG
Constructing DAG

Diagram showing a Delaunay triangulation with points labeled P₀, P₋₁, P₋₂ and corresponding triangles identified with labels from Δ₁ to Δ₁₆.
Constructing DAG
Point location

1. Start at the root of D,
2. Check the three children of the root and descend to the corresponding child,
3. check the children of this node, descend to a child whose triangle contains $p_r$,
4. until we reach a leaf of D, this leaf corresponding to a triangle in the current triangulation that contains $p_r$. 

How to choose $p_{-1}$ and $p_{-2}$?

and

How to implement the test of whether an edge is legal?
The first issue
The first issue
The first issue
The first issue

\[ L_{-1} \]

\[ L_{-2} \]

\[ p_{-2} \]

\[ p_{-1} \]

\[ p_0 \]
The first issue
The first issue

Position of a point \( p_j \) with respect to the oriented line from \( p_i \) to \( p_k \):

- \( p_j \) lies to the left of the line from \( p_i \) to \( p_{-1} \);
- \( p_j \) lies to the left of the line from \( p_{-2} \) to \( p_i \);
- \( p_j \) is lexicographically larger than \( p_i \).

By our choice of \( p_{-1} \) and \( p_{-2} \), the above conditions are equivalent.
Let $\overline{p_ip_j}$ be the edge of to be tested, and let $p_k$ and $p_l$ be the other vertices of the triangles incident to $\overline{p_ip_j}$ (if they exist).

- $\overline{p_ip_j}$ is an edge of the triangle $p_0p_{-1}p_{-2}$. These edges are always legal.
- The indices $i,j,k,l$ are all non-negative. ← this case is normal
- All other cases $\overline{p_ip_j}$ is legal if and only if $\min(k,l) < \min(i,j)$
Lemma 9.11:
The expected number of triangles created by the algorithm is at most $9n + 1$.

Proof.
$P_r := \{p_1, p_2, \ldots, p_r\}$
$Dg_r := Dg(\{p_{-2}, p_{-1}, p_0\} \cup P_r)$

$\#(\text{new triangles in step } r) \leq 2k - 3$
$k = \text{deg}(p_r, Dg_r)$
The Analysis

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$(\text{new triangles in step } r) \leq 2k - 3$
$k = \text{deg}(p_r, Dg_r)$
degree of \( p_r \), over all possible permutations of the set \( P \)?

\[ \text{Backwards analysis:} \]

- By Theorem 7.3: \# Edges in \( Dg_r \leq 3(r + 3) - 6 \)
- Total degree of the vertices in \( P_r < 2[3(r + 3) - 9] = 6r \)
- The expected degree of a random point of \( P_r \) is \( 6 \)
- We can bound the number of triangles created in step \( r \): 
  \[ E\left[ \sum_{\triangle \text{in step } r}\right] \leq 2 \cdot \text{deg}(p_r, Dg_r) - 3 \]
  \[ = 2 \cdot \text{deg}(p_r, Dg_r) - 3 \]
  \[ \leq 2 \cdot 6 - 3 = 9 \]
The Analysis

degree of $p_r$, over all possible permutations of the set $P$?

**Backwards analysis:**

- By Theorem 7.3: $\|\text{Edges in } Dg_r \| \leq 3(r + 3) - 6$.
- Total degree of the vertices in $P_r < 2[3(r + 3) - 9] = 6r$.
- The expected degree of a random point of $P_r \leq 6$.
- We can bound the number of triangles created in step $r$:
  
  $E[\|\text{\# \ Triangles in step } r\|] \leq E[2\text{\degree}(p_r, Dg_r) - 3]$
  
  $= 2E[\text{\degree}(p_r, Dg_r)] - 3$
  
  $\leq 2 \times 6 - 3 = 9$.
degree of $p_r$, over all possible permutations of the set $P$?

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- The expected degree of a random point of $P_r \leq 6$

$E[\# (\triangle s in step r)] \leq E[2\text{deg}(p_r, Dg_r) - 3]$

$$= 2E[\text{deg}(p_r, Dg_r)] - 3$$

$$\leq 2 \times 6 - 3 = 9$$
The Analysis

degree of $p_r$, over all possible permutations of the set $P$?

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- The expected degree of a random point of $P_r \leq 6$
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$$E[\# (\triangle s \text{ in step } r)] \leq E [2\text{deg}(p_r, Dg_r) - 3]$$

$$= 2E[\text{deg}(p_r, Dg_r)] - 3$$

$$\leq 2 \times 6 - 3 = 9$$

Total number of $\triangle s$ is at most $9n + 1$
degree of $p_r$, over all possible permutations of the set $P$?

Backwards analysis:

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- Total degree of the vertices in $P_r < 2[3(r + 3) - 9] = 6r$
- The expected degree of a random point of $P_r \leq 6$
- we can bound the number of triangles created in step $r$:

$$E[\# (\triangle s \text{ in step } r)] \leq E[2deg(p_r, Dg_r) - 3]$$

$$= 2E[deg(p_r, Dg_r)] - 3$$

$$\leq 2 \times 6 - 3 = 9$$

Total number of $\triangle s$ is at most $9n + 1$
The Analysis

Lemma 9.12:
The Delaunay triangulation can be computed in $O(n \log n)$ expected time, using $O(n)$ expected storage.

Proof.

- Space follows from nodes in $D$ representing triangles created, which by the previous lemma is $O(n)$.
- Not counting the time for point location,
- the creation of each triangle takes $O(1)$ time,
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- Let $K(\triangle) \subset P$ be the points inside the circumcircle of a given triangle $\triangle$
- Therefore the total time for the point location steps is:
  $$O(n + \sum_{\triangle} \text{card}(K(\triangle)))$$
- $\sum_{\triangle} \text{card}(K(\triangle)) = O(n \log n)$?
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Lemma 9.13:

If $P$ is a point set in general position, then

$$\sum_{\triangle} \text{card}(K(\triangle)) = O(n\log n)$$

Proof.

- $P$ is in general position, then every subset $P_r$ is in general position
- triangulation after insert $p_r$ is the unique triangulation $Dg_r$
- $T_r :=$ the set of $\triangle$s of $Dg_r$
- $T_r \setminus T_{r-1} =$ the set of Delaunay $\triangle$s created in stage $r$. (by definition)
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The Analysis

Rewrite the sum:

\[ \sum_{r=1}^{n} \left( \sum_{\triangle \in T_r \setminus T_{r-1}} \text{card}(K(\triangle)) \right) \]

- Let \( k(P_r, q) = \# \text{ of triangles } \triangle \in T_r \; ; q \in K(\triangle) \)
- Let \( k(P_r, q, p_r) = \# \text{ of triangles } \triangle \in T_r \; ; q \in K(\triangle) \; , p_r \text{ is incident to } \triangle \)
- so we have:

\[ \sum_{\triangle \in T_r \setminus T_{r-1}} \text{card}(K(\triangle)) = \sum_{q \in P \setminus P_r} k(P_r, q, p_r) \]

- But \( E \left[ k(P_r, q, p_r) \right] \leq ? \)
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The Analysis

- fix $P_r$, so $k(P_r, q, p_r)$ depends only on $p_r$
- Probability that $p_r$ is incident to a triangle is $3/r$
- Thus:
  $$E[k(P_r, q, p_r)] \leq \frac{3k(P_r, q)}{r}$$

- Using:
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- Any of the remaining \( n-r \) points is equally likely to appear as \( p_{r+1} \)
- So:
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The Analysis

- $k(P_r, p_{r+1})$, the number of triangles of $T_r$ that contain $p_{r+1}$
- These are the triangles that will be destroyed when $p_{r+1}$ is inserted. Theorem 9.6 (i)
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The Analysis

- By Theorem 9.1: $T_m$ has $2(m + 3) - 2 - 3 = 2m + 1$
- $T_{m+1}$ has two triangles more than $T_m$
- Thus, $\text{card}(T_r \setminus T_{r+1})$
  $\leq \text{card}(\text{triangles destroyed by } p_{r+1})$
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Remember we fixed \( P_r \) earlier

Consider all \( P_r \) by averaging over both sides of the inequality, but the inequality comes out identical.

\[
E[\# \text{ of triangles created by } p_r] = E[\# \text{ of edges incident to } p_{r+1} \text{ in } T_{r+1}] \leq 6
\]

Therefore:

\[
E[\sum_{\Delta \in T_r \setminus T_{r-1}} \text{card}(K(\Delta))] \leq 12\left(\frac{n-1}{r}\right)
\]

If we sum this over all \( r \), we have shown that:

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\sum_{\Delta} \text{card}(K(\Delta)) = O(n \log n)
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- Therefore:
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And thus, the algorithm runs in $O(n\log n)$ time.
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polyhedral terrain made! 😊