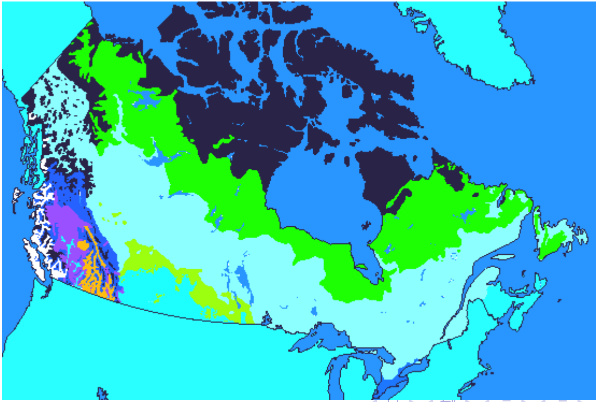
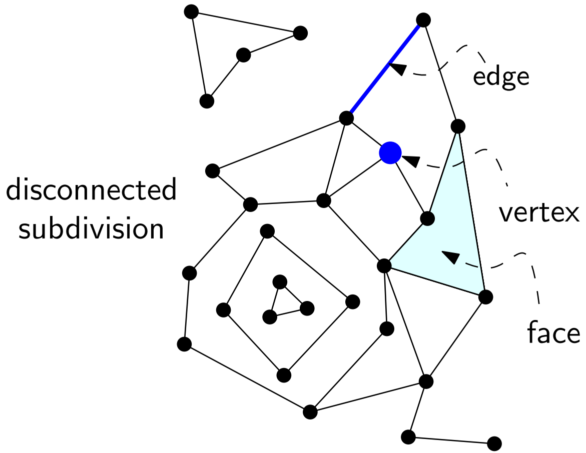
Computing the Overlay of Two Subdivisions

1392-1

- We have solved the easiest case of the map overlay problem, where the two maps are networks represented as collections of line segments.
- In general, maps have a more complicated structure: they are subdivisions of the plane into labeled regions.

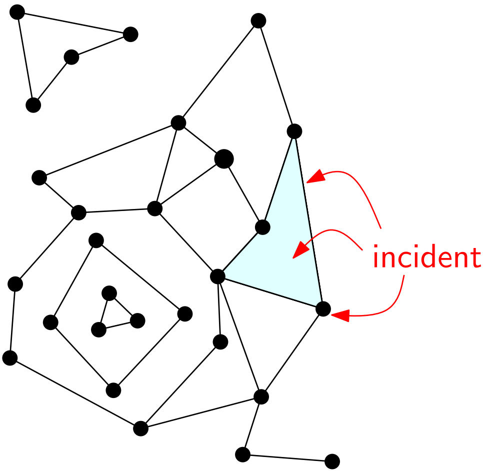


- Before we can give an algorithm for computing the overlay of two subdivisions, we must develop a suitable representation for a subdivision.
- Storing a subdivision as a collection of line segments is not such a good idea.
- Operations like reporting the boundary of a region would be rather complicated.
- Add topological information: which segments bound a given region, which regions are adjacent, and so on.



Complexity of a subdivision

#faces+#edges+#vertices.



Complexity of a subdivision

#faces+#edges+#vertices.

What kind of queries?

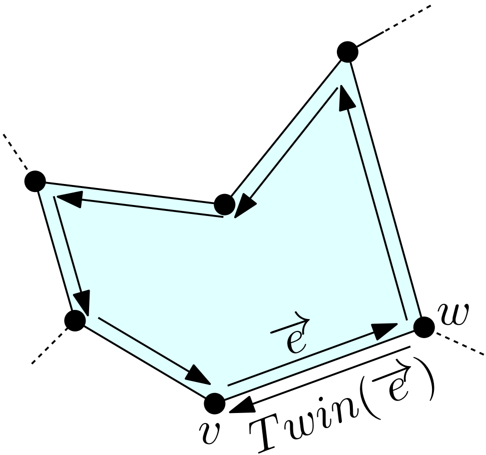
- What is the face containing a given point? (TOO MUCH!)
- Walking around the boundary of a given face,
- Find the face from an adjacent one if we are given a common edge,
- Visit all the edges around a given vertex.

The representation that we shall discuss supports these operations. It is called the doubly-connected edge list (DCEL).

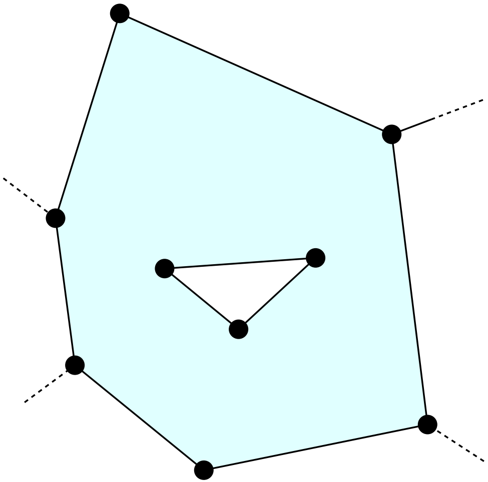
DCEL

DCEL contains:

- a record for each edge,
- a record for each vertex,
- a record for each face,
- plus attribute information.



To be able to traverse the boundary of a face, we need to keep a pointer to a half-edge of any boundary component and isolated points.



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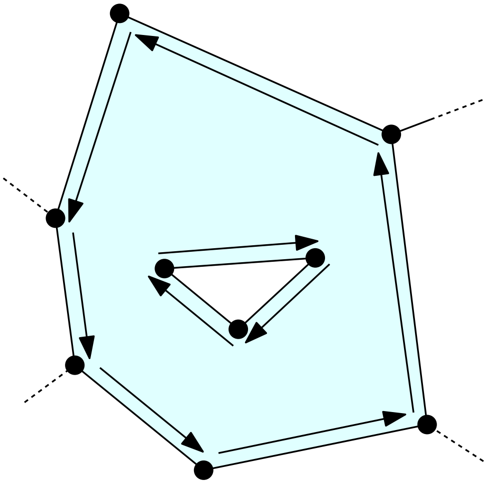

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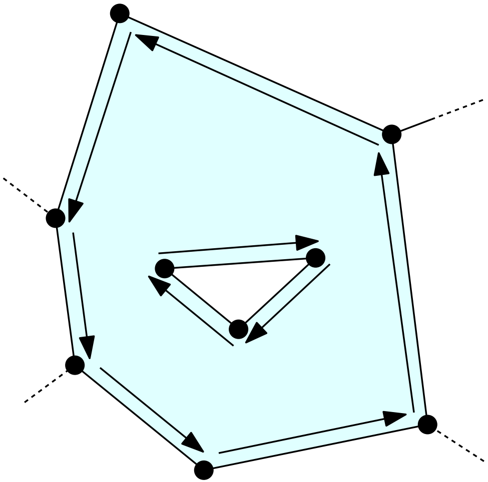

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DCEL contains:

- a record for each vertex,
  1. $Coordinates(v)$: the coordinates of v,
  2. $IncidentEdge(v)$: a pointer to an arbitrary half-edge that has v as its origin.
- a record for each face,
  1. $OuterComponent(f)$: to some half-edge on its outer boundary (nil if unbounded),
  2. $InnerComponents(f)$: a pointer to some half-edge on the boundary of the hole, for each hole.
- a record for each half-edge $\overrightarrow{e}$,
  1. $Origin(\overrightarrow{e})$: a pointer to its origin,
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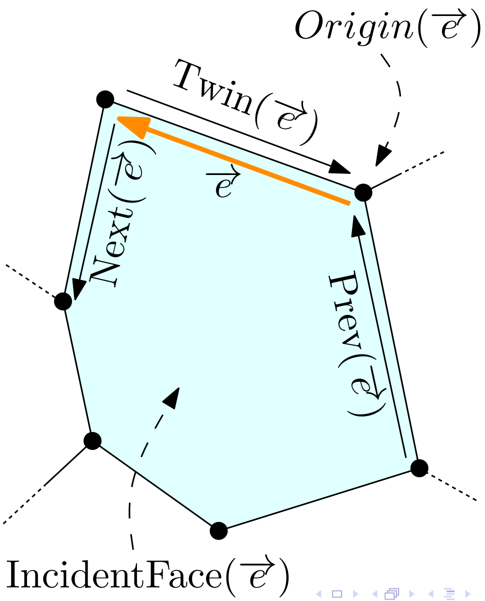
DCEL



Yazd Univ.

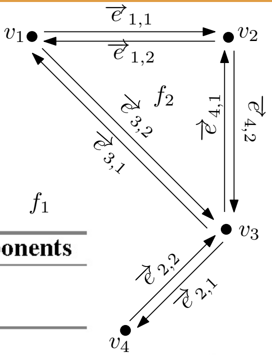
Computational Geometry

Doubly Connected Edge List (DCEL)

Computing the Overlay of Two Subdivisions

DCEL:Example

Vertex	Coordinates	IncidentEdge
v_1	$(0,4)$	$\vec{e}_{1,1}$
v_2	$(2,4)$	$\vec{e}_{4,2}$
v_3	$(2,2)$	$\vec{e}_{2,1}$
v_4	$(1,1)$	$\vec{e}_{2,2}$



Face	OuterComponent	InnerComponents
f_1	**nil**	$\vec{e}_{1,1}$
f_2	$\vec{e}_{4,1}$	**nil**

Half-edge	Origin	Twin	IncidentFace	Next	Prev
$\vec{e}_{1,1}$	v_1	$\vec{e}_{1,2}$	f_1	$\vec{e}_{4,2}$	$\vec{e}_{3,1}$
$\vec{e}_{1,2}$	v_2	$\vec{e}_{1,1}$	f_2	$\vec{e}_{3,2}$	$\vec{e}_{4,1}$
$\vec{e}_{2,1}$	v_3	$\vec{e}_{2,2}$	f_1	$\vec{e}_{2,2}$	$\vec{e}_{4,2}$
$\vec{e}_{2,2}$	v_4	$\vec{e}_{2,1}$	f_1	$\vec{e}_{3,1}$	$\vec{e}_{2,1}$
$\vec{e}_{3,1}$	v_3	$\vec{e}_{3,2}$	f_1	$\vec{e}_{1,1}$	$\vec{e}_{2,2}$
$\vec{e}_{3,2}$	v_1	$\vec{e}_{3,1}$	f_2	$\vec{e}_{4,1}$	$\vec{e}_{1,2}$
$\vec{e}_{4,1}$	v_3	$\vec{e}_{4,2}$	f_2	$\vec{e}_{1,2}$	$\vec{e}_{3,2}$
$\vec{e}_{4,2}$	v_2	$\vec{e}_{4,1}$	f_1	$\vec{e}_{2,1}$	$\vec{e}_{1,1}$

Time complexity of queries?

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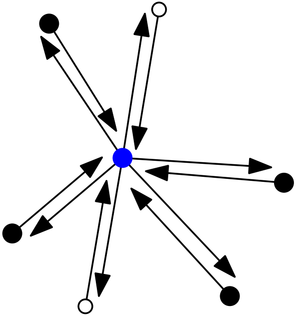

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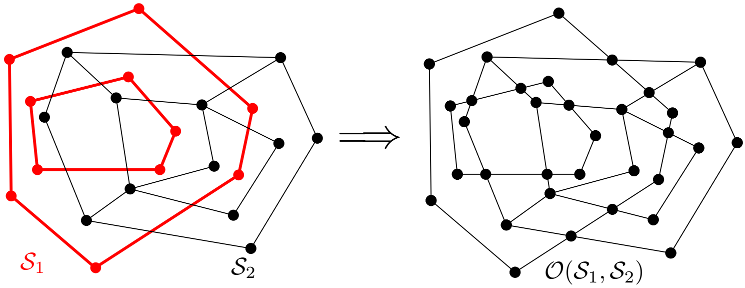

Main Idea in computing $\mathcal{O}(\mathcal{S}_1, \mathcal{S}_2)$

- Copy DCEL of $\mathcal{S}_1$ and $\mathcal{S}_2$ into new DCEL (Not a Valid DCEL).
- Make the new DCEL valid.

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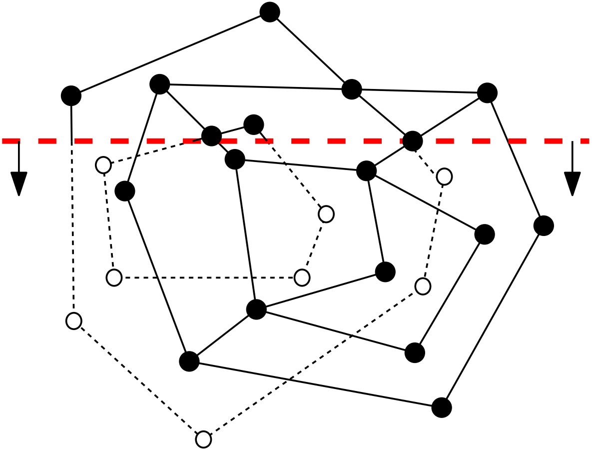
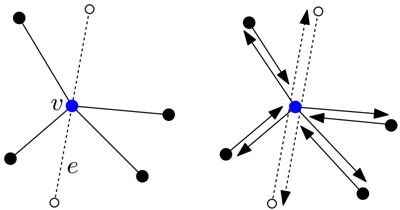


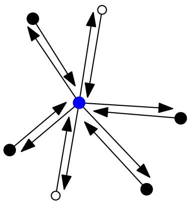
Computing the Overlay of Two Subdivisions

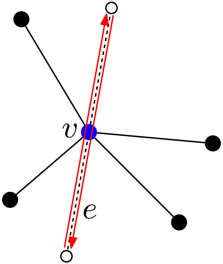
Updating half-edges

the geometric situation and the two doubly-connected edge lists before handling the intersection

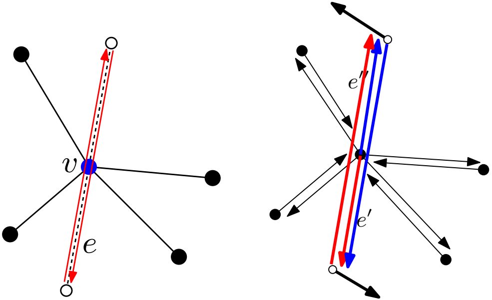


the doubly-connected edge list after handling the intersection





- Make two new edge with origin v.
- Set $Twin$ of new edges.
- Set $Next()$ of the two new half-edges.
- Set $Prev()$ of the half-edges to which these pointers point.



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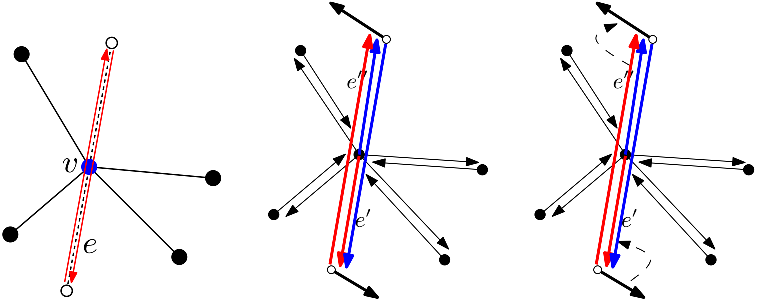

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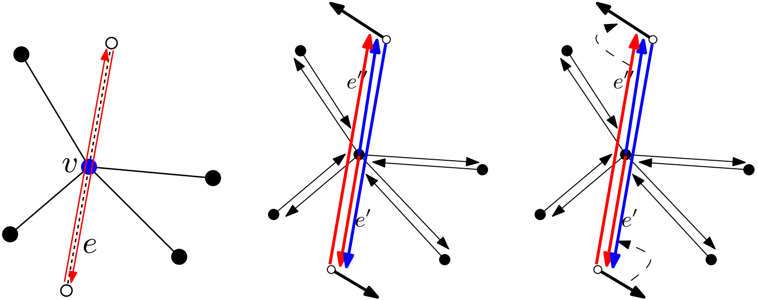

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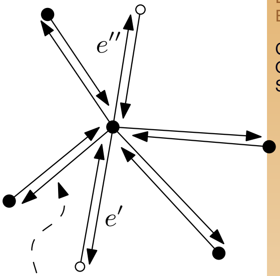
Fix the situation around v

- The half-edge for e' that has v as its destination must be linked to the first half-edge, seen clockwise from e', with v as its origin.
- The half-edge for e' with v as its origin must be linked to the first counterclockwise half-edge with v as its destination.
- The same for e''.
- Time complexity: $\mathcal{O}(m)$ (m: degree of v).



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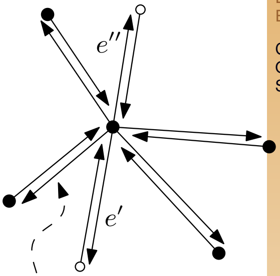

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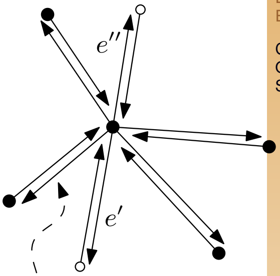

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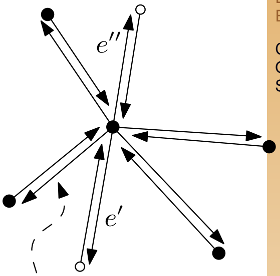
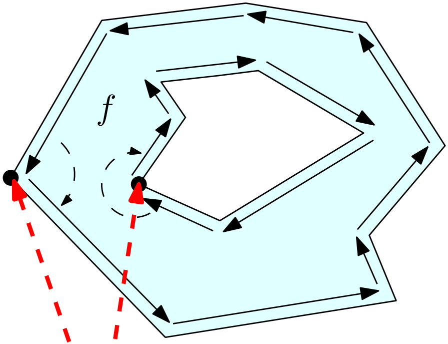


first clockwise half-edge from e' with v as its origin

Updating Faces:

- Create a face record for each $f \in \mathcal{O}(\mathcal{S}_1, \mathcal{S}_2)$.
- Set $OuterComponent(f)$ and $InnerComponent(f)$.

- # faces= # outer boundaries +1 (unbounded face).
- From half-edges we can construct the boundaries.
- To determine weather the boundary is outer boundary or boundary of a hole:



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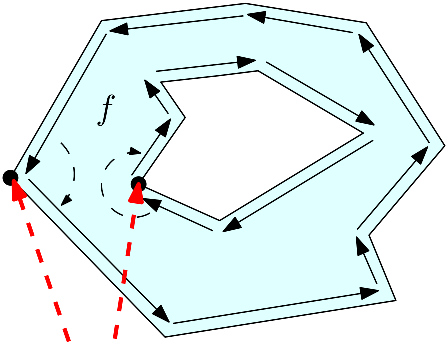

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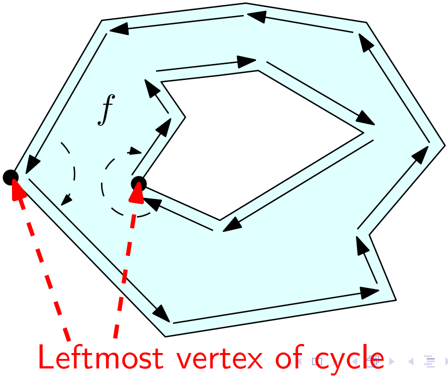

Which boundary cycles bound the same face?

- Construct a graph $\mathcal{G}$.
- Every boundary cycle is a node in $\mathcal{G}$.
- One node for the imaginary outer boundary of the unbounded face.
- Add an arc between two cycles if and only if one of the cycles is the boundary of a hole and the other cycle has a half-edge immediately to the left of the leftmost vertex of that hole cycle.
- If there is no half-edge to the left of the leftmost vertex of a cycle, then the node representing the cycle is linked to the node of the unbounded face.

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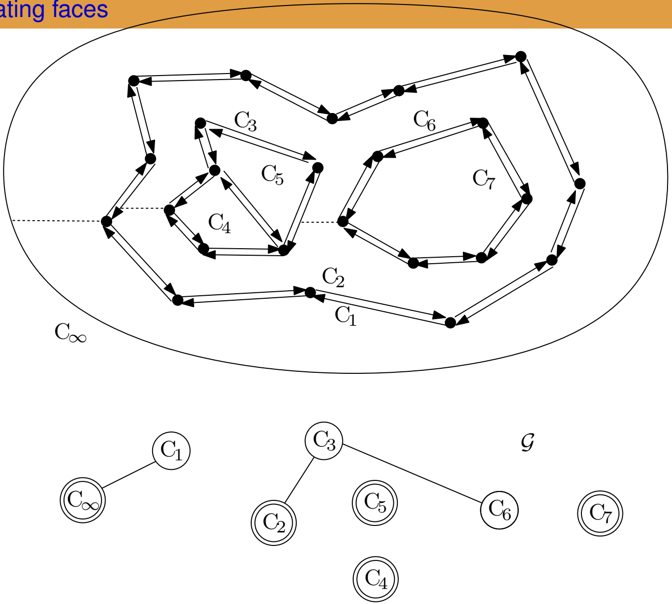

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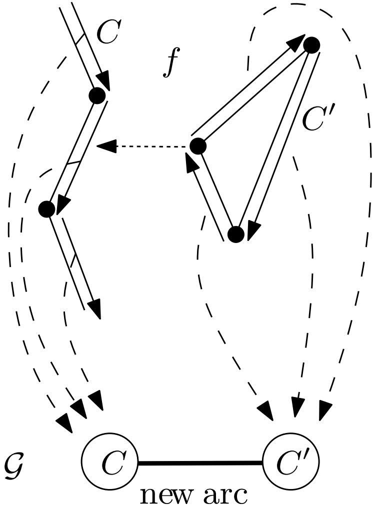
Updating faces

Lemma 2.5

Each connected component of the graph $\mathcal{G}$ corresponds exactly to the set of cycles incident to one face.


C
f
C'
$\mathcal{G}$
C
C'
new arc

Computing the Overlay of Two Subdivisions

Theorem 2.6

Let $\mathcal{S}_1$ be a planar subdivision of complexity n_1, let $\mathcal{S}_2$ be a subdivision of complexity n_2, and let $n := n_1 + n_2$. The overlay of $\mathcal{S}_1$ and $\mathcal{S}_2$ can be constructed in $\mathcal{O}(n \log n + k \log n)$ time, where k is the complexity of the overlay.

Application:Boolean Operations